On the Bootstrap for Spatial Econometric Models

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Introduction

- Bootstrap: Estimating the distribution of an estimator or test statistic by resampling one's data: treating the data as if they were the population.
- Its approximations can be at least as good as those from the first-order asymptotic theory.
  Useful when evaluating the asymptotic distribution is difficult.
- It can often be more accurate than the first-order asymptotic theory, i.e., asymptotic refinements.
- Bias correction, confidence intervals, hypothesis testing, etc.

Bootstrap Example

- $x \sim i.i.d.(\mu, \sigma^2)$
- data: $x_1, x_2, \ldots, x_n$.
- Statistic of interest: $\hat{\mu} = \frac{x_1+x_2+\cdots+x_n}{n}$.
- Bootstrap:
  draw $x_1^{(1)}, x_2^{(1)}, \ldots, x_n^{(1)}$, compute $\hat{\mu}^{(1)} = \frac{x_1^{(1)}+x_2^{(1)}+\cdots+x_n^{(1)}}{n}$;
  draw $x_1^{(2)}, x_2^{(2)}, \ldots, x_n^{(2)}$, compute $\hat{\mu}^{(2)} = \frac{x_1^{(2)}+x_2^{(2)}+\cdots+x_n^{(2)}}{n}$;
  ...
  draw $x_1^{(s)}, x_2^{(s)}, \ldots, x_n^{(s)}$, compute $\hat{\mu}^{(s)} = \frac{x_1^{(s)}+x_2^{(s)}+\cdots+x_n^{(s)}}{n}$.
  Approximate $\hat{\mu}$'s distribution by $\hat{\mu}^{(1)}, \ldots, \hat{\mu}^{(s)}$.

Asymptotic Refinements

Use some expansions, e.g., Edgeworth expansions.

- $$G(x) = P(T \le x) = \Phi(x) + n^{-1/2} q(x)\phi(x) + O(n^{-1}),$$
  $$\hat{G}(x) = P(T^* \le x|\mathcal{X}) = \Phi(x) + n^{-1/2}\hat{q}(x)\phi(x) + O_p(n^{-1}).$$

  Normal approximation: $G(x) - \Phi(x) = O(n^{-1/2})$;
  Bootstrap approximation:

  $$G(x) - \hat{G}(x) = n^{-1/2}[q(x) - \hat{q}(x)]\phi(x) + O_p(n^{-1}) = O_p(n^{-1}).$$

- $$G(x) = P(T \le x) = \Phi(\frac{x}{\sigma}) + n^{-1/2} q(\frac{x}{\sigma})\phi(\frac{x}{\sigma}) + O(n^{-1}),$$
  $$\hat{G}(x) = P(T^* \le x|\mathcal{X}) = \Phi(\frac{x}{\hat{\sigma}}) + n^{-1/2}\hat{q}(\frac{x}{\hat{\sigma}})\phi(\frac{x}{\hat{\sigma}}) + O_p(n^{-1}).$$
  $$G(x) - \hat{G}(x) = \Phi(\frac{x}{\sigma}) - \Phi(\frac{x}{\hat{\sigma}}) + O_p(n^{-1}) = O_p(n^{-1/2}).$$

Existence of Asymptotic Expansions

- Hall (1997): Smooth function model: X_i's are i.i.d. with mean μ, $A(\bar{X}) = [g(\bar{X}) - g(\mu)]/h(\mu)$ or $A(\bar{X}) = [g(\bar{X}) - g(\mu)]/h(\bar{X})$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i$.
- Götze and Hipp (1983, 1994): X_i's weekly dependent.

The Bootstrap for Spatial Econometric Models

- Many informal discussions and Monte Carlo Studies:
  - Spatial autoregressive (SAR) models: Anselin (1988, 1990), Can (1992).
  - Spatial moving average models: Fingleton (2008), Fingleton and Le Gallo (2008).
  - Moran's I under heteroskedasticity and non-normality: Lin et al. (2011).
  - Size distortions: Fingleton and Burridge (2010) and Burridge (2012) (Spatial J tests), Yang (2011) (LM tests).
  - Bias and robust variance: Su and Yang (2008), Yang (2012).
- No theoretical results on the validity.
- Existing results cannot be applied.

Objectives

- Consistency: Provides a consistent estimator of a statistic's asymptotic distribution.
  Provide a general proposition showing that the bootstrap for some statistics is consistent and provide some applications.
- Asymptotic refinements.

Bootstrap method

A general model—SARAR Model:

$y_n = \lambda W_n y_n + X_n\beta + u_n,\ u_n = \rho M_n u_n + \epsilon_n,\ \epsilon_{ni}$'s i.i.d. $\sim (0, \sigma^2)$.

- $(X_{ni}, y_{ni}) \Rightarrow ((W_n y_n)_i, X_{ni}, y_{ni})$.
- Residual bootstrap:
  - Estimate to derive $\hat{\theta}_n = (\hat{\lambda}_n, \hat{\beta}_n', \hat{\rho}_n)'$ and residuals $\hat{\epsilon}_n$.
  - Sample from the recentered residuals $(I_n - l_n l_n'/n)\hat{\epsilon}_n$ to derive ϵ_n^*, and generate pseudo data $y_n^* = (I_n - \hat{\lambda}_n W_n)^{-1}[X_n\hat{\beta}_n + (I_n - \hat{\rho}_n M_n)^{-1}\epsilon_n^*]$.
  - Use y_n^* to estimate θ and compute test statistics.

Linear-quadratic Forms in Spatial Econometric Models

- Linear-quadratic (LQ) forms of disturbances: The leading order terms of many estimators and test statistics. $[\epsilon_n' A_n \epsilon_n - \sigma_0^2 \operatorname{tr}(A_n) + b_n' \epsilon_n]/\sqrt{n}$.
- Estimators: the derivatives of the corresponding criterion function evaluated at the true parameter vector are often LQ forms.
  - MLE. $\sqrt{n}(\hat{\theta}_n - \theta_0) = \left(-\frac{1}{n} \mathrm{E} \frac{\partial^2 L_n(\theta_0)}{\partial\theta\partial\theta'}\right)^{-1} \frac{1}{\sqrt{n}} \frac{\partial L_n(\theta_0)}{\partial\theta} + o_P(1)$, where

$$L_n(\theta) = -\frac{n}{2}\ln(2\pi) - \frac{n}{2}\ln\sigma^2 + \ln|S_n(\lambda)| + \ln|R_n(\rho)| - \frac{1}{2\sigma^2}[S_n(\lambda)y_n - X_n\beta]' R_n'(\rho) R_n(\rho)[S_n(\lambda)y_n - X_n\beta], \quad (1)$$

with $S_n(\lambda) = I_n - \lambda W_n$ and $R_n(\rho) = I_n - \rho M_n$.

The reduced form of y_n: $y_n = S_n^{-1}(\lambda_0)[X_n\beta_0 + R_n^{-1}(\rho_0)\epsilon_n]$.

More Estimators

- GMME.
  Let $g_n(\gamma) = \frac{1}{n}(\epsilon_n'(\gamma)D_{1n}\epsilon_n(\gamma), \ldots, \epsilon_n'(\gamma)D_{mn}\epsilon_n(\gamma), \epsilon_n'(\gamma)F_n')'$, where $\epsilon_n(\gamma) = (I_n - \rho M_n)[(I_n - \lambda W_n)y_n - X_n\beta]$ and $\mathrm{tr}(D_{jn}) = 0$.

  $\min g_n'(\gamma)a_n a_n' g_n(\gamma) \Rightarrow$

$$\begin{aligned}&\sqrt{n}(\hat{\gamma}_n - \gamma_0)\\ &= -\Big(\mathrm{E}\,\frac{\partial g_n'(\gamma_0)}{\partial\gamma}a_n a_n'\,\mathrm{E}\,\frac{\partial g_n(\gamma_0)}{\partial\gamma'}\Big)^{-1}\Big(\mathrm{E}\,\frac{\partial g_n'(\gamma_0)}{\partial\gamma}\Big)a_n a_n'\sqrt{n}g_n(\gamma_0) + o_P(1).\end{aligned}$$

- The generalized spatial 2SLSE (Kelejian and Prucha, 1998): only linear instruments are used.

LQ Forms and Test Statistics

- Classical tests:
  - The likelihood framework: Wald, likelihood ratio and Lagrangian multiplier tests.
  - The GMM framework: Wald, distance and gradient tests.
- The Moran I test:

$$\frac{n}{\ell_n' M_n \ell_n} \frac{\hat{\epsilon}_n' M_n \hat{\epsilon}_n}{\hat{\epsilon}_n' \hat{\epsilon}_n}.$$

For the SE model $y_n = X_n\beta + u_n, u_n = \rho M_n u_n + \epsilon_n$,
or the SMA model $y_n = X_n\beta + u_n, u_n = \rho M_n \epsilon_n + \epsilon_n$,
the LM test statistic is

$$\frac{n}{\sqrt{\mathrm{tr}(M_n^2 + M_n' M_n)}} \frac{\hat{\epsilon}_n' M_n \hat{\epsilon}_n}{\hat{\epsilon}_n' \hat{\epsilon}_n} = \frac{n}{\sqrt{\mathrm{tr}(M_n^2 + M_n' M_n)}} \frac{\epsilon_n' H_n M_n H_n \epsilon_n}{\epsilon_n' H_n \epsilon_n}$$

$$= \frac{n}{\sqrt{\mathrm{tr}(M_n^2 + M_n' M_n)}} \frac{\epsilon_n' H_n M_n H_n \epsilon_n - \sigma_0^2 \,\mathrm{tr}(M_n H_n)}{(n - k_x)\sigma_0^2} + o_P(1),$$

where $H_n = I_n - X_n(X_n' X_n)^{-1} X_n'$.

LQ Forms and Test Statistics

- Generalized Moran's I (Kelejian and Prucha, 2001).
  Model $g_{i,n}(z_n, \theta_0) = u_{i,n}$.
  Test statistic: $(\hat{u}_n' W_n \hat{u}_n)/\hat{\sigma}_{Q_n}$.
  Assume that $n^{-1/2}\hat{u}_n' W_n \hat{u}_n = n^{-1/2}(\epsilon_n' A_n \epsilon_n + b_n' \epsilon_n) + o_P(1)$.
- Spatial J tests (Kelejian 2008, Kelejian and Piras 2011).
  $H_0: \quad y_n = \lambda_1 W_{1n} y_n + X_{1n}\beta_1 + u_{1n},\ u_{1n} = \rho_1 M_{1n} u_{1n} + \epsilon_{1n},$
  $H_1: \quad y_n = \lambda_2 W_{2n} y_n + X_{2n}\beta_2 + u_{2n},\ u_{2n} = \rho_2 M_{2n} u_{2n} + \epsilon_{2n},$
  $R_{1n}(\hat{\rho}_{1n}) y_n = \lambda_1 R_{1n}(\hat{\rho}_{1n}) W_{1n} y_n + R_{1n}(\hat{\rho}_{1n}) X_{1n}\beta_1 + \alpha R_{1n}(\hat{\rho}_{1n})\hat{y}_n + \epsilon_n.$
- Cox-type tests (Jin and Lee 2013):
  $L_{2n}(\hat{\theta}_{2n}) - L_{1n}(\hat{\theta}_{1n}) - \hat{\mathrm{E}}[L_{2n}(\hat{\theta}_{2n}) - L_{1n}(\hat{\theta}_{1n})].$

Features of a LQ Form

- w.l.o.g., assume that A_n is symmetric.
- $[\epsilon_n' A_n \epsilon_n - \sigma_0^2 \operatorname{tr}(A_n) + b_n' \epsilon_n]/\sqrt{n} =$ $\frac{1}{\sqrt{n}}[2\sum_{i=2}^n \sum_{j=1}^{i-1} a_{n,ij}\epsilon_{ni}\epsilon_{nj} + \sum_{i=1}^n a_{n,ii}^2(\epsilon_{ni}^2 - \sigma_0^2) + \sum_{i=1}^n b_{ni}\epsilon_{ni}]$.
- U-statistic: for n, $g_n(x_1, \ldots, x_n) = avg\, f(x_{\varphi(1)}, \ldots, x_{\varphi(r)})$, where $\varphi(1), \varphi(r) \in \{1, 2, \ldots, n\}$.
- Smooth function model: X_i's are i.i.d. with mean μ, $A(\bar{X}) = [g(\bar{X}) - g(\mu)]/h(\mu)$ or $A(\bar{X}) = [g(\bar{X}) - g(\mu)]/h(\bar{X})$, where $\bar{X} = \frac{1}{n}\sum_{i=1}^n X_i$.

Consistency: Setup

- t_n: asymptotically standard normal.
- $t_n = t_n(\hat{\theta}_n, \theta_0, \hat{\eta}_n, \epsilon_n)$.
- Bootstrapped t_n: $t_n^* = t_n(\hat{\theta}_n^*, \hat{\theta}_n, \hat{\eta}_n^*, \epsilon_n^*)$.
- $t_n = c_n/\sigma_{c_n} + d_n = c_n/\sigma_{c_n} + o_P(1)$, where $c_n = n^{-1/2}[\epsilon_n' A_n \epsilon_n - \sigma_0^2 \operatorname{tr}(A_n) + b_n' \epsilon_n]$ and $\sigma_{c_n}^2 = \mathrm{E}\, c_n^2 = n^{-1}\Big[2\sigma_0^4 \operatorname{tr}(A_n^2) + \sigma_0^2 b_n' b_n + \sum_{i=1}^n \Big((\mu_4 - 3\sigma_0^4) a_{n,ii}^2 + 2\mu_3 a_{n,ii} b_{ni}\Big)\Big]$.
- $c_n^* = n^{-1/2}[\epsilon_n^{*\prime} A_n \epsilon_n^* - \sigma_n^{*2} \operatorname{tr}(A_n) + b_n' \epsilon_n^*]$ with $\sigma_n^{*2} = \frac{1}{n} \sum_{i=1}^n \epsilon_{ni}^{*2}$.
- $d_n^* = t_n^* - c_n^*/\sigma_{c_n}^*$.

Assumptions

Kelejian and Prucha (2001): Central limit therem for a LQ form.

1. The ϵ_{ni}'s in $\epsilon_n = (\epsilon_{n1}, \ldots, \epsilon_{nn})'$ are i.i.d. $(0, \sigma_0^2)$ and $\mathrm{E}\,|\epsilon_{ni}|^{4(1+\delta)} < \infty$ for some $\delta > 0$.
2. The sequence of symmetric matrices $\{A_n\}$ are bounded in both row and column sum norms, and elements of the vectors $\{b_n\}$ satisfy $\sup_n n^{-1} \sum_{i=1}^n |b_{ni}|^{2(1+\delta)} < \infty$.
3. The $\sigma_{c_n}^2$ is bounded away from zero.

Consistency: A General Result

Under Assumptions 1–3,

$$\sup_{x\in\mathbb{R}} \left| \mathrm{P}(c_n/\sigma_{c_n} \le x) - \Phi(x) \right| \le r_n,$$

where $r_n = K\sigma_{c_n}^{-2(1+\delta)/(3+2\delta)} n^{-\delta/(3+2\delta)} \Big((K_a+1)^{1+2\delta} \big(K_a \, \mathrm{E}\,|\epsilon_{ni}^2 - \sigma_0^2|^{2+2\delta} + 2^{2+2\delta} K_a (\mathrm{E}\,|\epsilon_{ni}|^{2+2\delta})^2 + K_b \, \mathrm{E}\,|\epsilon_{ni}|^{2+2\delta} \big) + 4^{1+\delta} \big(\sigma_0^4 K_a^4 (\mu_4 - \sigma_0^4) + 4\sigma_0^8 K_a^4 + \sigma_0^2 K_a^2 (\mu_3^2 K_a + \sigma_0^4 K_b)(K_a+1) + 2|\mu_3|\sigma_0^2 K_a^3 (|\mu_3| K_a + \sigma_0^2 K_b) \big)^{(1+\delta)/2} \Big)^{1/(3+2\delta)}$.

$$\sup_{x\in\mathbb{R}} \left| \mathrm{P}^*(c_n^*/\sigma_{c_n}^* \le x) - \Phi(x) \right| \le r_n^*.$$

Consistency: A General Result

$$\sup_{x\in\mathbb{R}}\left|\mathrm{P}^*\big(c_n^*/\sigma_{c_n}^* + d_n^* \le x\big) - \mathrm{P}\big(c_n/\sigma_{c_n} + d_n \le x\big)\right|$$
$$\le r_n + \mathrm{P}(|d_n| > \tau_n) + r_n^* + \mathrm{P}^*(|d_n^*| > \tau_n) + 2^{1/2}\pi^{-1/2}\tau_n,$$

where τ_n is any positive term depending only on n, and e_n is a nonstochastic term depending on n.

If $d_n = O_P(n^{-1/2})$, we may let $\tau_n = kn^{-\alpha}$ with $\alpha < 1/2$. It remains to show that $\mathrm{P}^*(|d_n^*| > \tau_n) = o_P(1)$.

$$\sup_{x\in\mathbb{R}}\left|\mathrm{P}^*\Big(\big(c_n^*/\sigma_{c_n}^* + d_n^*\big)e_n^* \le x\Big) - \mathrm{P}\Big(\big(c_n/\sigma_{c_n} + d_n\big)e_n \le x\Big)\right|$$
$$\le r_n + \mathrm{P}(|d_n| > \tau_n) + r_n^* + \mathrm{P}^*(|d_n^*| > \tau_n)$$
$$+ 2^{1/2}\pi^{-1/2}\tau_n + \sup_{x\in\mathbb{R}}\left|\Phi(x/e_n) - \Phi(x/e_n^*)\right|.$$

Proof

Write c_n as $c_n = \sum_{i=1}^{n} c_{ni}$ with

$$c_{ni} = n^{-1/2}\Big(a_{n,ii}(\epsilon_{ni}^2 - \sigma_0^2) + 2\epsilon_{ni}\sum_{j=1}^{i-1} a_{n,ij}\epsilon_{nj} + b_{ni}\epsilon_{ni}\Big).$$

Consider the σ-fields $\mathscr{F}_{n0} = \{\varnothing, \Omega\}$, $\mathscr{F}_{ni} = \sigma(\epsilon_{n1}, \ldots, \epsilon_{ni})$, $1 \leq i \leq n$, where Ω is the sample space. Then $\{c_{ni}, \mathscr{F}_{ni}, 1 \leq i \leq n, n \geq 1\}$ forms a martingale difference array.

Proof

Theorem (Heyde and Brown, 1970): If there is a constant δ with $0 < \delta \leq 1$ such that $\mathrm{E}\,|c_{ni}|^{2+2\delta} < \infty$, then there exists a finite constant K depending only on δ, such that

$$\sup_x \left| \mathrm{P}(c_n \leq \sigma_{c_n} x) - \Phi(x) \right| \leq$$

$$K \left\{ \sigma_{c_n}^{-2-2\delta} \left(\sum_{i=1}^{n} \mathrm{E}\,|c_{ni}|^{2+2\delta} + \mathrm{E} \left| \left(\sum_{i=1}^{n} \mathrm{E}(c_{ni}^2 | \mathscr{F}_{n,i-1}) \right) - \sigma_{c_n}^2 \right|^{1+\delta} \right) \right\}^{1/(3+2\delta)} \qquad (\star)$$

$$= K(T_{n1} + T_{n2})^{1/(3+2\delta)}.$$

Thus if

$$\lim_{n\to\infty} T_{n1} = 0 \text{ and } \lim_{n\to\infty} T_{n2} = 0,$$

$\mathrm{P}(c_n \leq \sigma_{c_n} x)$ converges uniformly to $\Phi(x)$ and a bound on the rate of convergence is given by $(\star)$.

Consistency: Moran's I

- When $\epsilon_n \sim N(0, \sigma_0^2 I_n)$,

$$I_n = \frac{n}{\sqrt{\mathrm{tr}(M_n^2 + M_n' M_n)}} \frac{\hat{\epsilon}_n' M_n \hat{\epsilon}_n}{\hat{\epsilon}_n' \hat{\epsilon}_n} = \frac{n}{\sqrt{\mathrm{tr}(M_n^2 + M_n' M_n)}} \frac{\epsilon_n' H_n M_n H_n \epsilon_n}{\epsilon_n' H_n \epsilon_n}.$$

  $\sup_{x \in \mathbb{R}} |\mathrm{P}^*(\mathbb{I}_n^* \leq x) - \mathrm{P}(\mathbb{I}_n \leq x)| = o_P(1)$.
- When ϵ_{ni}'s are not normal,

$$I_n' = \frac{\hat{\epsilon}_n' M_n \hat{\epsilon}_n}{\sqrt{n} \hat{\sigma}_{c_n}},$$

  where
  $\hat{\sigma}_{c_n}^2 = n^{-1}(\hat{\mu}_{4n} - 3\hat{\sigma}_n^4) \sum_{i=1}^n (H_n M_n H_n)_{ii}^2 + n^{-1} \hat{\sigma}_n^4 \mathrm{tr}[H_n M_n H_n (M_n + M_n')]$.

  $\sup_{x \in \mathbb{R}} |\mathrm{P}^*(\mathbb{I}_n'^* \leq x) - \mathrm{P}(\mathbb{I}_n' \leq x)| = o_P(1)$.

Consistency: Spatial J Tests

$$H_0: \quad y_n = \lambda_1 W_{1n} y_n + X_{1n}\beta_1 + u_{1n}, \quad u_{1n} = \rho_1 M_{1n} u_{1n} + \epsilon_{1n},$$
$$H_1: \quad y_n = \lambda_2 W_{2n} y_n + X_{2n}\beta_2 + u_{2n}, \quad u_{2n} = \rho_2 M_{2n} u_{2n} + \epsilon_{2n}.$$

Estimate the model
$R_{1n}(\hat{\rho}_{1n}) y_n = \lambda_1 R_{1n}(\hat{\rho}_{1n}) W_{1n} y_n + R_{1n}(\hat{\rho}_{1n}) X_{1n}\beta_1 + \alpha R_{1n}(\hat{\rho}_{1n})\hat{y}_n + \epsilon_n$, and test whether $\alpha = 0$.

- Kelejian and Piras (2011): Spatial 2SLS.
- GMM estimation:
  $g_n(\gamma) = \frac{1}{n}(\epsilon_n'(\gamma) D_{1n}\epsilon_n(\gamma), \ldots, \epsilon_n'(\gamma) D_{mn}\epsilon_n(\gamma), \epsilon_n'(\gamma) F_n')'$.
- Lemma 1: $n^{1/2}(\hat{\theta}_n - \theta_0) = O_P(1)$ and
  $n^{1/2}(n^{-1}\sum_{i=1}^n \hat{\epsilon}_{ni}^r - \mathrm{E}\,\epsilon_{ni}^r) = O_P(1)$.
- Lemma 2: $\mathrm{P}^*(n^a \|\hat{\theta}_n^* - \hat{\theta}_n\| > \eta) = o_P(1)$ and
  $\mathrm{P}^*(n^a | n^{-1}\sum_{i=1}^n \hat{\epsilon}_{ni}^{*r} - \mathrm{E}^*\,\epsilon_{ni}^{*r}| > \eta) = o_P(1)$ for $\eta > 0$ and $0 \le a < 1/2$.

Asymptotic Refinements and Asymptotic Expansions

- Edgeworth expansions: $\mathrm{P}(t_n \leq \eta) = \sum_{i=0}^{\infty} n^{-i/2} g_n(\eta)$.
- Existing results: a smooth function of sample averages of independent random vectors and/or stationary dependent random vectors.
- A LQ form: Cannot be written as simple sample averages of disturbances or their cross-products.
- No result on the Edgeworth expansions of a LQ form.
- Two cases:
  - When $\epsilon_n \sim N(0, \sigma_0^2 I_n)$, directly establish Edgeworth expansions.
  - When ϵ_{ni}'s are not normal, establish an asymptotic expansion based on martingales.

Edgeworth Expansion: Normal Disturbances

When $\epsilon_n \sim N(0, \sigma_0^2 I_n)$,

$$\sup_{x \in \mathbb{R}} \left| \mathrm{P}(c_n/\sigma_{c_n} \le x) - [\Phi(x) + \kappa_n(1 - x^2)\Phi^{(1)}(x)] \right| = O(n^{-1}),$$

$$\sup_{x \in \mathbb{R}} \left| \mathrm{P}^*(c_n^*/\sigma_{c_n}^* \le x) - [\Phi(x) + \kappa_n^*(1 - x^2)\Phi^{(1)}(x)] \right| = O_P(n^{-1}),$$

where $\kappa_n = n^{-3/2}\sigma_{c_n}^{-3}[4\sigma_0^6 \operatorname{tr}(A_n^3)/3 + \sigma_0^4 b_n' A_n b_n] = O(n^{-1/2})$ with $\sigma_{c_n}^2 = n^{-1}[2\sigma_0^4 \operatorname{tr}(A_n^2) + \sigma_0^2 b_n' b_n]$ and $\kappa_n^* = n^{-3/2}\sigma_{c_n}^{*-3}[4\sigma_n^{*6} \operatorname{tr}(A_n^3)/3 + \sigma_n^{*4} b_n' A_n b_n] = O_P(n^{-1/2})$ with $\sigma_{c_n}^{*2} = n^{-1}[2\sigma_n^{*4} \operatorname{tr}(A_n^2) + \sigma_n^{*2} b_n' b_n]$, and for $r \ge 3$, there exist real polynomials $P_{n3}(x), \dots, P_{nr}(x)$ with bounded coefficients such that

$$\sup_{x \in \mathbb{R}} \left| \mathrm{P}(c_n/\sigma_{c_n} \le x) - \Phi(x) - \Phi^{(1)}(x) \sum_{i=3}^{r} n^{-(i-2)/2} P_{ni}(x) \right| = O(n^{-(r-1)/2}).$$

Proof

- The characteristic function of c_n/σ_{c_n} can be derived, as $\epsilon_n \sim N(0, \sigma_0^2 I_n)$. Let $\mathrm{E}\exp(itc_n/\sigma_{c_n}) = \exp(g_n(t) - t^2/2)$.
- The orders of $g_n(t)$'s derivatives can also be derived: $g_n(0) = g_n^{(1)}(0) = g_n^{(2)}(0) = 0$, $|g_n^{(k)}(t)| \leq \frac{c_{k5}(\iota_n \sigma_0^2)^{k-2}}{n^{(k-2)/2}\sigma_{c_n}^{k-2}}$ for $k \geq 3$.
- A smoothing inequality in Feller (1970), for all $T > 0$,

$$\sup_{x \in \mathbb{R}} |\mathrm{P}(c_n/\sigma_{c_n} \leq x) - (\Phi(x) - \kappa_n \Phi^{(3)}(x))| \leq \frac{1}{\pi}\int_{-T}^{T} \left|\frac{\varphi_n(t) - \gamma_n(t)}{t}\right| dt + \frac{24\sup_x |\Phi^{(1)}(x) - \kappa_n \Phi^{(4)}(x)|}{\pi T}.$$

- let $T = n\sigma_{c_n}^2$. $|t| \leq \frac{\sqrt{2n}\sigma_{c_n}}{8\iota_n \sigma_0^2}$ and $\frac{\sqrt{2n}\sigma_{c_n}}{8\iota_n \sigma_0^2} < |t| \leq T$.

Asymptotic refinements: Moran's I

- Under regularity conditions, $\mathrm{P}^*(\mathbb{I}_n^* \leq x) - \mathrm{P}(\mathbb{I}_n \leq x) = O_P(n^{-1})$.
- $\mathrm{P}\left(\frac{\epsilon_n' A_n \epsilon_n}{\epsilon_n' B_n \epsilon_n} \leq \eta\right) = \mathrm{P}\big(\epsilon_n'(A_n - \eta B_n)\epsilon_n \leq 0\big)$, where B_n is psd.
- Alternatively, $\frac{\epsilon_n' A_n \epsilon_n}{\epsilon_n' B_n \epsilon_n} = \frac{\epsilon_n' A_n \epsilon_n}{\mathrm{E}(\epsilon_n' B_n \epsilon_n)} - \frac{\epsilon_n' A_n \epsilon_n [\epsilon_n' B_n \epsilon_n - \mathrm{E}(\epsilon_n' B_n \epsilon_n)]}{\epsilon_n' B_n \epsilon_n \, \mathrm{E}(\epsilon_n' B_n \epsilon_n)}$.
- The delta method: If $S_n = T_n + O_P(n^{-i/2})$, then $\mathrm{P}(S_n \leq x) = \mathrm{P}(T_n \leq x) + O(n^{-i/2})$ in general.
- When ϵ_{ni}'s are not normal,

$$I_n' = \frac{\hat{\epsilon}_n' M_n \hat{\epsilon}_n}{\sqrt{n}\hat{\sigma}_{c_n}},$$

where
$\hat{\sigma}_{c_n}^2 = n^{-1}(\hat{\mu}_{4n} - 3\hat{\sigma}_n^4)\sum_{i=1}^n (H_n M_n H_n)_{ii}^2 + n^{-1}\hat{\sigma}_n^4 \, \mathrm{tr}[H_n M_n H_n (M_n + M_n')]$.
Other statistics....

Asymptotic Expansion: Non-normal Disturbances

- Non-normal disturbances: a closed form characteristic function is no longer available.
- Mykland (1993): One-term asymptotic expansions for martingales.
- "The Edgeworth expansion for martingales".

Asymptotic Expansion: Non-normal Disturbances

Under regularity conditions,

$$\int_{-\infty}^{+\infty} h(x)\, dF_n(x) = \int_{-\infty}^{+\infty} h(x)\, d\Phi(x) + \frac{1}{6} n^{-1/2}\, \mathrm{E}\Big[\big(\psi_o(Y) + 2\psi_p(Y)\big) h^{(2)}(Y)\Big] + o(n^{-1/2}), \qquad (*)$$

where $F_n(x) = \mathrm{P}(c_n/\sigma_{c_n} \leq x)$, Y is the normal random variable that c_n/σ_{c_n} converges to, and $\psi_o(Y)$ and $\psi_p(Y)$ are linear function of Y, uniformly on a set ℓ of functions h which are twice differentiable, with h, $h^{(1)}$ and $h^{(2)}$ uniformly bounded, and with $\{h^{(2)}, h \in \ell\}$ being equicontinuous a.e. Lebesgue. Denote the convergence in $(*)$ by $o_2(n^{-1/2})$ (Mykland, 1993), then

$$F_n(x) = \Phi(x) + \frac{1}{6} n^{-1/2}\Big(\psi_o^{(1)}(x) + 2\psi_p^{(1)}(x) - [\psi_o(x) + 2\psi_p(x)]x\Big)\Phi^{(1)}(x) + o_2(n^{-1/2}).$$

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- The expansion generally does not hold when h is an indicator function of an interval, so it is a "smoothed" expansion.
- When ϵ_{ni}'s are normal, it can be verified that $\frac{1}{6}n^{-1/2}[\psi_o^{(1)}(x) + 2\psi_p^{(1)}(x) - [\psi_o(x) + 2\psi_p(x)]x] = (1 - x^2)\lim_{n\to\infty} \kappa_n$.

Proof

- Integrability conditions.
  1. $\sum_{i=1}^{n} \mathrm{E}\, X_{i,n}^4 = O(n^{-1})$.
  2. $n^{1/2}[\sum_{i=1}^{n} \mathrm{E}(X_{i,n}^2 | \xi_{i-1,n}) - \mathrm{E}(X_{i,n}^2)]$ is uniformly integrable.
- Central limit theorem. $\left(\sum_{i=1}^{n} X_{i,n}, \sqrt{n}\left(\sum_{i=1}^{n}\left(X_{i,n}^2 - E(X_{i,n}^2 | \xi_{i-1,n})\right)\right), \sqrt{n}\left(\sum_{i=1}^{n}[\mathrm{E}(X_{i,n}^2 | \xi_{i-1,n}) - E(X_{i,n}^2)]\right)\right)$ is asymptotically trivariate normal.
- Martingale CLT: If $\sum_{i=1}^{n} \mathrm{E}\,|X_{i,n}|^{2+\delta} \to 0$ for some $\delta > 0$, and $\sum_{i=1}^{n} \mathrm{E}(X_{i,n}^2 | \xi_{i-1,n}) \xrightarrow{p} 1$, then $\sum_{i=1}^{n} X_{i,n} \xrightarrow{d} N(0, 1)$.
- The Cramér-Wold device.
- No application yet.

Summary and Concluding Remarks

- Many estimators and test statistics in spatial econometric models can be studied based on LQ forms.
- The bootstrap is in general consistent for statistics that can be approximated by LQ forms.
- For asymptotic refinements, we establish the Edgeworth expansion for LQ forms with normal disturbances, and an asymptotic expansion based on martingales for LQ forms with non-normal disturbances.
- Some tests are based on the asymptotic normality of a vector of LQ forms, say, chi-square tests. Results on a vector of LQ forms are needed.