The λ-nucleolus for NTU games and Shapley procedure

Chih Chang[*]
Department of Mathematics
National Tsing Hua University
Hsinchu, Taiwan

Peng-An Chen
Department of Mathematics
National Taitung University
Taitung, Taiwan

April 25, 2011

Abstract

We first scrutinize several possible extensions of the nucleolus for NTU games. Among them, the λ-nucleolus is recommended. Shapley(1969) proposed a procedure, referred to Shapley procedure, to extend the Shapley value to the non-transferable utility value for NTU games. It is known that the procedure can be applied to a variety of solutions. In particular, the λ-nucleolus is the nucleolus version derived by the procedure. Then we try to argue that Shapley procedure might provide a good extension if one tries to extend a solution for TU games to NTU cases.

Keywords: TU games, the nucleolus, NTU games, the λ-nucleolus, continuity, upper semi-continuity, Shapley procedure

[*]The authors are grateful to Mamoru Kaneko for many helpful comments and discussions on both substantive matters and exposition which much improve the paper. They are also grateful to Yan-An Hwang and Hans Peters for helpful comments.

1 Introduction

To define a solution for games without transferable utilities (NTU games), one usually employs the ideas of solutions for games with transferable utilities (TU games). Two criteria for such an extension are considered: (A) *the coincidence property*: it coincides with its original version on TU games, and (B) *the preservation criterion:* nice properties of its original version can be preserved.

First, we scrutinize several possible extensions of the nucleolus for NTU games. It is known that the nucleolus for TU games satisfies *symmetry*, *continuity*, and *the core property,* where the core property means that a solution is contained in the core if it is non-empty, please see Schmeidler(1969) for these facts. Here we will specify criterion (B) to be the three properties. *An impossibility Theorem* is presented to illustrate that there is no solution for NTU games satisfying criterion (B). It means that some nice things disappear during the process of extension.

Among several extensions that we study, the λ-*nucleolus* is recommended because it satisfies (A) and (B) to a large extend. Moreover, the λ-*excess function* which is used to derive the λ-nucleolus for NTU games is axiomatized. It is the unique function satisfying *excess preservation*, *excess additivity,* and *excess invariance.*

Shapley(1969) proposed a procedure, referred to *Shapley procedure*, to extend the Shapley value to the non-transferable utility value(the NTU value) for NTU games. It is known that the procedure can be applied to a variety of solutions. In particular, the λ-nucleolus is the nucleolus extension derived by the approach.

Second, we argue that Shapley procedure might be able to provide a good extension if one tries to extend a solution for TU games to NTU cases. Aumann(1985) took advantage of a specific relationship between Shapley value and the NTU value to axiomatize the NTU value. In fact, every pair of solutions, a solution for TU games and its corresponding solution for NTU games derived via the procedure, share the relationship. Therefore, Aumann's idea can be used to axiomatize all this type of solutions for NTU games straightforwardly. Due to the relationship too, we show that a solution for NTU games derived via the procedure is upper semi-continuous(*u.s.c.*) if its corresponding solution for TU games is u.s.c., in particular, the λ-nucleolus is u.s.c. Both results are crucial for a solution concept. Further studies of this type of solutions for NTU games are needed.

The results of the paper are classified to be four parts presented from section 3 to section 6.

In section 3, we scrutinize two existing extensions of the nucleolus by Kalai(1975) and Nakayama(1983) , respectively. An *non-continuity Theorem* is presented to show that both Kalai's and Nakayama's versions are not u.s.c.

In section 4, an impossibility Theorem is presented to show that there is no solution for NTU games satisfying the three properties of (B).

In section 5, we first define the λ-excess function and axiomatize it. Then the λ-nucleolus is defined and properties of it are studied.

In section 6, we study upper semi-continuity and axiomatization of the class of solutions for NTU games derived via Shapley procedure.

2 Definitions and concepts

In this section, we introduce some basic definitions and concepts to be discussed in this paper, in particular, TU games, NTU games, and two well-known solution concepts: the core and the nucleolus. Several properties for solutions are defined too.

Denote $N = \{1, 2, \cdots, n\}$ to be the *player set and* $e^N = (1, 1, ..., 1) \in \mathbb{R}^N$. An non-empty subset S of N is called a *coalition* and $\mathcal{P}$ is the set of coalitions of N. Given $x, y \in \mathbb{R}^N$, we denote $x \ll y$ if $x_i < y_i$ $\forall i \in N$, $x < y$ if $x_i \leq y_i$ $\forall i \in N$ and $x \neq y$, $x \leq y$ if $x_i \leq y_i$ $\forall i \in N$. Given $S \in \mathcal{P}$, let $x(S) = \Sigma_{i \in S} x_i$ and x_S the *restriction* of x on $\mathbb{R}^S$, where

$$\mathbb{R}^S = \left\{x \in \mathbb{R}^N : x_i = 0 \text{ if } i \notin S\right\}.$$

A *TU game* is a pair (N, v), where $v : 2^N \to \mathbb{R}$ with $v(\emptyset) = 0$. For convenience, we denote (N, v) simply by the small letter v if no confusion arises. Denote $\mathbf{G}_N$ to be the set of all TU games with the player set N.

Given a game $v \in \mathbf{G}_N$, the *pre-imputation set* of v is

$$I(v) = \{x \in \mathbb{R}^N : x(N) = v(N) \},$$

and the *imputation set* of v is

$$I^*(v) = \{x \in \mathbb{R}^N : x(N) = v(N) \text{ and } x_i \geq v(\{i\}) \text{ for } i \in N \}.$$

Each element in $I(v)$ is called a *payoff vector*.

A *solution* σ defined on $\mathbf{G}'_N \subseteq \mathbf{G}_N$ is a mapping such that $\sigma(v) \subseteq I(v)$ for all $v \in \mathbf{G}'_N$.

The *core* of v is defined by

$$C(v) = \{x \in I(v) \mid x(S) \geq v(S) \text{ for all } S \in \mathcal{P}\}.$$

To define the nucleolus of v, we need various concepts. The *excess* e is a real-valued function defined on $\mathbb{R}^N \times \mathcal{P} \times \mathbf{G}_N$ such that, given $x \in \mathbb{R}^N$, $S \in \mathcal{P}$, and $v \in \mathbf{G}_N$,

$$e(x, S, v) = v(S) - x(S).$$

It is interpreted to be a measure of "dissatisfaction" of coalition S at x. We reshuffle the set $\{e(x, S, v) : S \in \mathcal{P}\}$ in the descending way, i.e., $e(x, S_k, v) \geq e(x, S_{k+1}, v)$ for all $k = 1, 2, \cdots, 2^n - 2$. The resulting vector is

$$\theta(x, v, e) = (e(x, S_1, v), e(x, S_2, v), \cdots, e(x, S_{2^n-1}, v)). \tag{1}$$

In a word, $\theta(x, v, e)$ orders the "dissatisfactions" of coalitions at x; the highest "dissatisfaction" first, the second-highest "dissatisfaction" second, and so forth.

If $x, y \in \mathbb{R}^N$, $\theta(x, v, e)$ is *lexicographically less than* $\theta(y, v, e)$, denoted by $\theta(x, v, e) < (lex)\, \theta(y, v, e)$, if and only if there is an integer k, $1 \leq k \leq 2^n - 1$, such that the first k-1 components of $\theta(x, v, e)$ and $\theta(y, v, e)$ are equal and the kth component of $\theta(x, v, e)$ is less than the kth component of $\theta(y, v, e)$. We denote $\theta(x, v, e) \leq (lex)\, \theta(y, v, e)$ if $\theta(x, v, e) < (lex)\, \theta(y, v, e)$ or $\theta(x, v, e) = \theta(y, v, e)$.

Given $v \in \mathbf{G}_N$, the *nucleolus* of v is defined to be

$$\mathcal{N}(v, e) = \{\, x \in I^*(v) \mid \theta(x, v, e) \leq (lex)\ \theta(y, v, e)\ \ \forall y \in I^*(v)\}. \tag{2}$$

Schmeidler(1969) proved that $\mathcal{N}(v, e)$ is single-valued and is non-empty.

Next, we will mention several properties that are relevant to the paper.

A solution σ satisfies *the core property* if $\sigma(v) \subseteq C(v)$, where $C(v) \neq \emptyset$. We say that σ is continuous: If $\{v_n\}_{n=1}^{\infty}$ is a sequence of games in $\mathbf{G}_N$ and $v_n \to v$, then $\mathcal{N}(v_n, e) \to \mathcal{N}(v, e)$ as $n \to \infty$. Denote π to be a permutation on N. We say that π is a symmetry of $v \in \mathbf{G}_N$ if $v(\pi S) = v(S)$ for every coalition S. σ is *symmetric* if $\sigma(v) = \pi(\sigma(v))\ \forall v \in \mathbf{G}_N$ and all symmetries π of v.

An *NTU game* is a pair (N, V), where V assigns each coalition S a proper subset of $\mathbb{R}^S$ satisfying

$(A1)$ $V(S)$ is closed, comprehensive, and non-empty $\forall S \in \mathcal{P}$;
$(A2)$ $IR(V) = \{x \in V(N) \mid x_i \geq y \ \forall \ y \in V(\{i\}) \text{ and } \forall \ i \in N \ \} \neq \emptyset$.

Conditions $(A2)$ is called *essentiality*. For convenience, we denote (N, V) simply by the capital letter V. Denote $\mathcal{G}_N$ to be the set of NTU games with the player set N. Each element in $IR(V)$ is called a payoff vector as well.

Given $v \in \mathbf{G}_N$ and $V \in \mathcal{G}_N$, if

$$V(S) = \left\{x \in \mathbb{R}^S : x(S) \leq v(S)\right\} \ \forall \ S \in \mathcal{P}, \tag{3}$$

we call v and V *corresponding* to each other. Hence, a TU game v can be viewed as a game in $\mathcal{G}_N$ and $V \in \mathcal{G}_N$ can be viewed as a TU game if it satisfies (3).

A *solution* σ defined on $\mathcal{G}_N$ is a mapping such that $\sigma(V) \subseteq V(N)$ for all $V \in \mathcal{G}_N$. σ is *non-empty* if $\sigma(V) \neq \emptyset$ for all $V \in \mathcal{G}_N$.

The core of a game $V \in \mathcal{G}_N$ is defined by

$$C(V) = \{\, z \in V(N) \mid \nexists \ y \in V(S) \text{ with } y_S \gg z_S \ \forall S \in \mathcal{P}\}.$$

A solution σ satisfies *the core property* if $\sigma(V) \subseteq C(V)$ if $C(V) \neq \emptyset$.

Given $S \in \mathcal{P}$, let $\mathcal{F}$ be the collection of non-empty closed subsets of $\mathbb{R}^S$. We denote $h_S : \mathcal{F} \times \mathcal{F} \longrightarrow \mathbb{R} \cup \{-\infty, +\infty\}$ to be the *extended Hausdorff metric*, please see Aliprantis and Border(2006). Given two games $V, W \in \mathcal{G}_N$, we define

$$\rho(V, W) = \max_{S \in \mathcal{P}} h_S(V(S), W(S)).$$

Let $\{V_n\}_{n=1}^{\infty}$ be a sequence of games in $\mathcal{G}_N$ and $V_n \to V$ in the topology induced by the extended Hausdorff metric. The solution σ *is u.s.c.* if $x_n \in \sigma(V_n) \ \forall \ n$ and $x_n \to x \in V(N)$ as $n \to \infty$, then $x \in \sigma(V)$.

The idea to define a symmetric solution for NTU games is the same as what we did on TU games. Given a permutation π on N and $x \in \mathbb{R}^S$, we denote $\pi(x) \in \mathbb{R}^{\pi(S)}$ to be $(\pi(x))_{\pi(i)} = x_i$ for every $i \in S$. We say that the permutation π is a symmetry of $V \in \mathcal{G}_N$ if $V(\pi S) = \pi V(S) = \{\pi(x) : x \in V(S)\} \ \forall S \in \mathcal{P}$. A solution σ is *symmetric* if $\sigma(V) = \pi(\sigma(V))$ $\forall V \in \mathcal{G}_N$ and all symmetries π of V.

3 Two extensions

Kalai(1975) and Nakayama(1983) each provided an extension of the nucleolus for NTU games. The purpose of the section is to scrutinize whether these two extensions satisfy (A) and (B). An non-continuity Theorem is presented to study continuity of the nucleolus for NTU games. We see that the one by Kalai satisfies the core property only and the other one by Nakayama violates u.s.c.

The idea of the nucleolus is to minimize "total dissatisfaction" among all feasible payoff vectors in the sense of lexicographic order and the excess is defined to measure "dissatisfaction" for TU games. To define the nucleolus for NTU games, it is natural to extend the excess to NTU games first. We call a real-valued function defined on $\mathbb{R}^N \times \mathcal{P} \times \mathcal{G}_N$ an *excess function* for NTU games.

Let E be an excess function for NTU games. Following (1) and (2), $\theta(x, V, E)$ is defined except that v and e are replaced by V and E, respectively. Then $\mathcal{N}(V, E)$, the nucleolus for $V \in \mathcal{G}_N$ derived by E, is defined to be

$$\mathcal{N}(V, E) = \{ x \in IR(V) \mid \theta(x, V, E) \leq (lex)\, \theta(y, V, E) \; \forall\, y \in IR(V) \}. \quad (4)$$

Given a set $D \subseteq \mathbb{R}^N$, let $int(D)$ be the interior of a set D, ∂D be the *boundary* of D, and $\partial^+ D = \partial D \cap \mathbb{R}^N_+$. Given $x \in \mathbb{R}^N$, $S \in \mathcal{P}$, and $A \subseteq \mathbb{R}^S$, we denote

$$G^S(x) = \left\{ z \in \mathbb{R}^S \,|\, z_i \leq x_i \text{ for every } i \in S \right\}$$

and $G^S(A) = \cup_{a \in A} G^S(a)$.

Let

$$IR^+(V) = \left\{ x \in \mathbb{R}^N \,|\, x_i \geq y_i \;\forall\, y \in V(\{i\}) \text{ and } \forall\, i \in N \right\} \neq \emptyset.$$

Theorem 1 (Non-continuity) *There is no excess function for NTU games E satisfying the following 3 conditions simultaneously:*

(i) E is continuous (or u.s.c.) in $x \in \mathbb{R}^N$ and $V \in \mathcal{G}_N$,

$$(ii)\; E(x, S, V) \begin{cases} = 0, & \text{if } x_S \in \partial V(S), \\ > 0, & \text{if } x_S \in int(V(S)), \\ < 0, & \text{if } x_S \notin V(S), \end{cases}$$

(iii) $\mathcal{N}(V, E)$ is u.s.c.

The idea to prove the result is to propose a class of 3-person games in which the required properties do not hold simultaneously.

Example 2 *Let $N = \{1, 2, 3\}$ and k be a large enough positive integer. Set*

$$\begin{aligned}
&A = (1, 1, 0),\ A_n = \left(1 + \tfrac{1}{k+n-2}, 1 + \tfrac{1}{k+n-2}, 0\right),\\
&B = (1, 0, 1),\ B_n = \left(1 + \tfrac{1}{k+n-2}, 0, 1 + \tfrac{1}{k+n-2}\right),\\
&C = C_n = (0, 1, 1),\\
&D = (0, 0, 1),\ D_n = \left(0, 0, 1 + \tfrac{1}{k+n-2}\right),\\
&E = \left(0, 1 - \tfrac{1}{k}, 1 - \tfrac{1}{k}\right),\ E_n = \left(0, 1 - \tfrac{1}{k}\left(1 - \tfrac{1}{n}\right), 1 - \tfrac{1}{k}\left(1 - \tfrac{1}{n}\right)\right),\\
&F = (0, 1, 0),\ \text{and } F_n = \left(0, 1 + \tfrac{1}{k+n-2}, 0\right).
\end{aligned}$$

Consider $V \in \mathcal{G}_N$ given by

$$\begin{aligned}
&V(\{i\}) = G^{\{i\}}(\{(0,0,0)\}) \text{ for } i = 1, 2, 3,\\
&V(\{1,2\}) = G^{\{1,2\}}(A),\ V(\{1,3\}) = G^{\{1,3\}}(B),\\
&V(\{2,3\}) = G^{\{2,3\}}(\overline{DE} \cup \overline{EF}),\ \text{and}\\
&V(N) = G^N\left(\left\{x \in \mathbb{R}^N \mid x_1 + x_2 + x_3 \le 2\right\} \cap \mathbb{R}_+^{\{1,2,3\}}\right).
\end{aligned}$$

For arbitrary positive integer n, let V_n be defined as

$$\begin{aligned}
V_n(\{i\}) &= G^{\{i\}}(\{(0,0,0)\}) \text{ for } i = 1, 2, 3,\\
V_n(\{1,2\}) &= G^{\{1,2\}}(A_n),\ V_n(\{1,3\}) = G^{\{1,3\}}(B_n),\\
V_n(\{2,3\}) &= G^{\{2,3\}}(\overline{D_n E_n} \cup \overline{E_n F_n}),\ \text{and}\\
V_n(N) &= G^N\left((\text{the hyperplane determined by } A_n, B_n, \text{ and } C) \cap \mathbb{R}_+^{\{1,2,3\}}\right)\\
&= G^N\left(\left\{x \in \mathbb{R}^N \mid \tfrac{k+n-3}{k+n-1}x_1 + x_2 + x_3 \le 2\right\} \cap \mathbb{R}_+^{\{1,2,3\}}\right)
\end{aligned}$$

Since $A_n \to A$, $B_n \to B$, $D_n \to D$, $E_n \to E$, and $F_n \to F$ as $n \to \infty$, we have $V_n \to V$ in the topology induced by extended Hausdorff metric. Please see Figure 1.

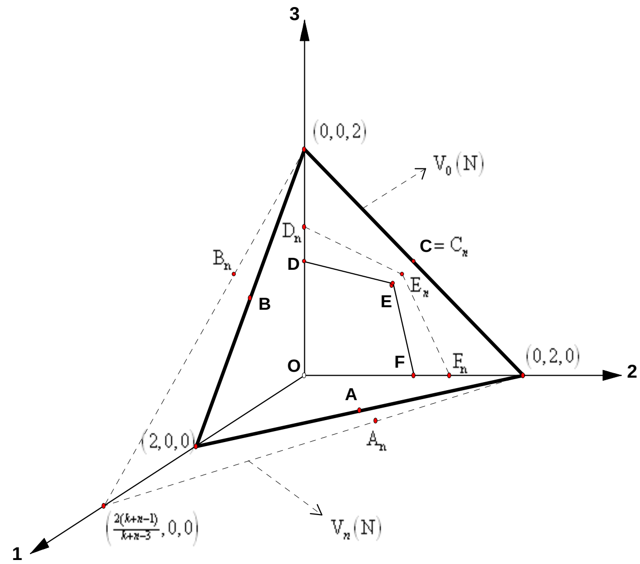

Figure 1.

Claim: *If* V *is a game,* $\mathcal{N}(V,E)$ *satisfies the core property.*

If $x \in V(N) - C(V)$, *there exist a* $S \in \mathcal{P}$ *and a* $y \in V(S)$ *such that* $y_S \gg x_S$, *that is,* $x \in int(V(S))$ *and* $E(x,S,V) > 0$ *by* (ii). *If* $x \in C(V)$, *then* $x_S \notin int(V(S))$ $\forall S \in \mathcal{P}$. *It implies that* $E(x,S,V) \le 0$ $\forall S \in \mathcal{P}$ *by* (ii) *again. We obtain that* $\mathcal{N}(V,E)$ *must be in* $C(V)$. *The Claim is obtained.*

We see that $C(V) = \{A,B,C\}$ *and* $C(V_n) = \{A_n, B_n\} \neq \emptyset$ *for all* n. *By Claim,* $\mathcal{N}(V_n,E) \subseteq C(V_n) \to \{A,B\}$ *as* $n \to \infty$. *To show that* $\mathcal{N}(V_n,E) \nrightarrow \mathcal{N}(V,E)$, *it is enough to prove that* $\mathcal{N}(V,E) = \{C\}$.

$E(A,S,V) = 0$ *if* $S \in \{\{3\},\{1,2\},\{1,3\},\{2,3\},\{1,2,3\}\}$ *and* $E(A,S,V) < 0$ *if* $S \in \{\{1\},\{2\}\}$ *by* (ii). *Hence,*

$$\theta(A,V,E) = (0,0,0,0,0,-a_1,-a_2), \text{ where } a_2 \ge a_1 > 0.$$

Similarly, we derive that

$$\theta(B,V,E) = (0,0,0,0,0,-a_3,-a_4), \text{ where } a_4 \ge a_3 > 0,$$

and

$$\theta(C,V,E) = (0,0,0,0,-a_5,-a_6,-a_7), \text{ where } a_7 \ge a_6 \ge a_5 > 0.$$

$\theta(C,V,E)$ *is strictly less than both* $\theta(A,V,E)$ *and* $\theta(B,V,E)$ *in lexicographical order. We obtain that* $\mathcal{N}(V,E) = \{C\}$.

We can view the games in the previous Example a specific class of n-person games. Indeed, players other than $N = \{1, 2, 3\}$ are dummy in the additive sense.

3.1 Kalai's extension

Kalai(1975) reformulated the properties of the excess into four conditions and defined the excess function for NTU games E_K as follows: Given $S \in \mathcal{P}$ and $V \in \mathcal{G}_N$,

(a) If $x, y \in \mathbb{R}^N$ and $x_S = y_S$, then $E_K(x, S, V) = E_K(y, S, V)$.

(b) If $x, y \in \mathbb{R}^N$ and $x_S \ll y_S$, then $E_K(x, S, V) > E_K(y, S, V)$.

(c) If $x \in \mathbb{R}^N$ and $x_S \in \partial V(S)$, then $E_K(x, S, V) = 0$.

(d) E_K is continuous in V and $x \in \mathbb{R}^N$.

He showed the following facts: (i) $\mathcal{N}(V, E_K)$, the nucleolus for $V \in \mathcal{G}_N$ derived by E_K, is non-empty and satisfies the core property. (ii) A specific E_K is proposed to demonstrate that $\mathcal{N}(V, E_K)$ is not u.s.c.

We would like to ask further. Is there an E_K such that the nucleolus derived by it is u.s.c.? The fact that there is no E_K such that $\mathcal{N}(V, E_K)$ is u.s.c. can be obtained by Non-continuity Theorem straightforwardly. Indeed, conditions (i) and (ii) of the theorem can be obtained by (d), (b), and (c).

Remark 3 *Hausdorff metric $\bar{h}$, please see Dugundji(1966), was employed to measure distance of two games in Kalai(1975). Hence, he required an NTU game satisfying not only $(A1)$ and $(A2)$ but also upper boundedness, that is, $\exists a \in \mathbb{R}^S$ such that $V(S) \subseteq G^S(a)$ $\forall S \in \mathcal{P}$.*

In fact, the result of Example 2 still holds if we use Hausdorff metric instead. Indeed, all games considered in the example are upper bounded.

Note that if $V_n \to V$ as $n \to \infty$ in the topology induced by Hausdorff metric, then V_n converges to V in the topology induced by extended Hausdorff metric as well.

The following example shows that $\mathcal{N}(V, E_K)$ does not satisfy the coincidence property.

Example 4 *Let $v \in \mathbf{G}_{\{1,2\}}$ such that $v(\{1\}) = v(\{2\}) = 0$ and $v(\{1, 2\}) = 1$. Then $\mathcal{N}(v, e) = \{(1/2, 1/2)\}$. Let V correspond to v. Then*

$$IR(V) = \{(x_1, x_2) : x(\{1, 2\}) \leq 1,\ x_i \geq 0, i = 1, 2\}.$$

Define the excess function E_K *to be*

$$E_K(x,\{1\},V)=v(\{1\})-x_1,\ E_K(x,\{2\},V)=2[v(\{2\})-x_2],\ \textit{and}$$
$$E_K(x,\{1,2\},V)=v(\{1,2\})-x(\{1,2\}).$$

E_K *satisfies conditions* (a) - (d) *and* $\mathcal{N}(V,E_K)=\{(2/3,1/3)\}\neq\mathcal{N}(v,e)$.

The following example shows that $\mathcal{N}(V,E_K)$ is not symmetric.

Example 5 *Let* $V\in\mathcal{G}_{\{1,2\}}$ *such that*

$$V(\{i\})=\{x\in\mathbb{R}:x\leq 0\}\ \textit{for}\ i=1,2,\ \textit{and}$$
$$V(\{1,2\})=G^{\{1,2\}}\{x\in\mathbb{R}_+^{\{1,2\}}:x(\{1,2\})\leq 1\}.$$

Define the excess function E_K *as follows.*

$$E_K(x,\{1\},V)=-2x_1,\ E_K(x,\{2\},V)=-x_2,\ \textit{and}$$
$$E_K(x,\{1,2\},V)=1-(x_1+x_2).$$

We see that E_K *satisfies* (a) - (d) *and* $\mathcal{N}(V,E_K)=\{(\frac{1}{3},\frac{2}{3})\}$.

Define $\pi:\{1,2\}\to\{1,2\}$ *by* $\pi(1)=2$ *and* $\pi(2)=1$. *Then* π *is a symmetry of* V *and* $\mathcal{N}(V,E_K)\neq\pi\mathcal{N}(V,E_K)$.

3.2 Nakayama's extension

Nakayama(1983) proposed an extension of the nucleolus by allowing the interpersonal utility comparison. The payoff vector is determined proportionally to a given vector of weights. He proved that the extension is non-empty and satisfies the core property. We are going to show that it satisfies the coincidence property and symmetry, but is not u.s.c.

Denote the set of vectors of weights to be

$$\Delta^N=\left\{w\in\mathbb{R}^N:\Sigma_{i\in N}w_i=1\text{ and }w_i\geq 0\text{ for all }i\in N\right\}.$$

Given $w\in\Delta^N$, $S\in\mathcal{P}$, and $V\in\mathcal{G}_N$, let

$$h(w,S)=\max\{h:h\cdot w_S\in V(S)\},$$

and let e_Y be a real-valued function defined on $\Delta^N\times\mathcal{P}\times\mathcal{G}_N$

$$e_Y(w,S,V)=\Sigma_{i\in S}(h(w,S)-h(w,N))\cdot w_i.$$

$h(w,S)\cdot w_S \in \partial V(S)$ is the payoff vector for S determined by w and $e_Y(w,S,V)$ means the difference between the sum of the payoffs of players in S determined by S and w and the sum of the payoffs of players in S determined by N and w.

Since Nakayama(1983) required a game V satisfying $\sup V(\{i\}) > 0 \ \forall\ i \in N$ and considered only individually rational payoff vectors, we define

$$\Delta^+(V) = \left\{ w \in \Delta^N : h(w,N)\cdot w_i \geq \sup V_{\{i\}} \text{ for all } i \in N \right\}.$$

Following (1), for each $w \in \Delta^+(V)$, let

$$\theta(w,V,e_Y) = (e_Y(w,S_1,V), e_Y(w,S_2,V), \cdots, e_Y(w,S_{2^n-1},V)),$$

where $S_k \in \mathcal{P}$ for all $k = 1,2,\cdots,2^n-1$ and $e_Y(w,S_k,V) \geq e_Y(w,S_{k+1},V)$ for all $k = 1,2,\cdots,2^n-2$.

Then an extension of the nucleolus for NTU games is defined to be, given a game V,

$$\mathcal{N}(V,e_Y) = \{ h(w,N)\cdot w \in IR(V) \mid \theta(w,V,e_Y) \leq (lex)\, \theta(w',V,e_Y) \ \forall\ w' \in \Delta^+(V)\}$$

$\mathcal{N}(V,e_Y)$ is non-empty and satisfies the core property.

Lemma 6 *Let $V \in \mathcal{G}_N$. If $x \in \partial V(N) \cap IR(V)$, there is a $w \in \Delta^+(V)$ such that $x(N) = h(w,N)$ and $x = w\cdot x(N) = w \cdot h(w,N)$.*

Proof. Set $w = \frac{x}{x(N)}$. Then $w \in \Delta^+(V)$ and $x = x(N)\cdot w \in IR(V) \cap \partial V(N)$. Since $h(w,N)\cdot w \in \partial V(N) \cap IR(V)$, we obtain that $x = w\cdot x(N) = w\cdot h(w,N)$ and $x(N) = h(w,N)$. ■

Given a game V and a $w \in \Delta^+(V)$, we denote

$$x_w = h(w,N)\cdot w \in IR(V) \cap \partial V(N). \qquad (5)$$

Together the observation with the Lemma 6, there is a one to one and onto mapping from $\Delta^+(V)$ to $IR(V) \cap \partial V(N)$.

Next, we are going to extend e_Y to be an excess function.

Given $x_w \in IR(V) \cap \partial V(N)$, $S \in \mathcal{P}$, and $V \in \mathcal{G}_N$, we define

$$e_Y(x_w,S,V) = \Sigma_{i\in S} h(w,S)\cdot w_i - (x_w)_i.$$

Then $e_Y(w,S,V) = e_Y(x_w,S,v)\ \forall w \in \Delta^+(V)$ by (5). Hence, instead of $\Delta^N \times \mathcal{P} \times \mathcal{G}_N$, the domain of e_Y can be viewed as $(IR(V) \cap \partial V(N)) \times \mathcal{P} \times \mathcal{G}_N$.

If V corresponds to v, then

$$\begin{aligned} e_Y(x_w,S,V) &= \Sigma_{i\in S}(h(w,S) - h(w,N)) \cdot w_i \\ &= v(S) - \Sigma_{i\in S} h(w,N) \cdot w_i \\ &= v(S) - x_w(S) = e(x_w,S,v). \end{aligned}$$

We obtain that $\mathcal{N}(V,e_Y)$ satisfies the coincidence property, that is, $\mathcal{N}(V,e_Y) = \mathcal{N}(v,e)$.

Let E_Y be a real-valued function defined on $IR^+(V) \times \mathcal{P} \times \mathcal{G}_N$. Given $x \in IR^+(V)$ and $S \in \mathcal{P}$,

$$E_Y(x,S,V) = \Sigma_{i\in S}\left(h\left(\frac{x}{x(N)},S\right) - x(N)\right) \cdot (\frac{x}{x(N)})_i.$$

We see that E_Y is continuous in x and V.

Lemma 7 *Given $V \in \mathcal{G}_N$, $S \in \mathcal{P}$, and $x \in IR^+(V)$,*

(1) *if $x \in \partial V(N) \cap IR^+(V)$, then $E_Y(x,S,V) = e_Y(x,S,V)$.*

(2)

$$E_Y(x,S,V) \begin{cases} = 0, & \text{if } x_S \in \partial V(S) \cap IR^+(V), \\ > 0, & \text{if } x_S \in int(V(S)) \cap IR^+(V), \\ < 0, & \text{if } x_S \notin V(S) \cap IR^+(V), \end{cases}$$

(3) $\mathcal{N}(V,E_Y) = \mathcal{N}(V,e_Y)$. *That is, $\mathcal{N}(V,E_Y)$ is non-empty.*

Proof. Let $x \in IR(V) \cap \partial V(N)$ and $w = \frac{x}{x(N)}$. Then, by Lemma 6,

$$\begin{aligned} E_Y(x,S,V) &= \Sigma_{i\in S}(h(w,S) - x(N)) \cdot w_i \\ &= \Sigma_{i\in S}(h(w,S) - h(w,N)) \cdot w_i = e_Y(x,S,V). \end{aligned}$$

(1) is obtained. (2) can be derived by using the same idea.

Suppose that there is a $y \in \mathcal{N}(V,E_Y) \cap int(V(N))$. Then there is a $y' \in IR(V) \cap \partial V(N)$ such that $y \ll y'$ and $\frac{y'}{y'(N)} = \frac{y}{y(N)}$. It follows that $E_Y(y,S,V) > E_Y(y',S,V)$ for every $S \in \mathcal{P}$, and hence, $\theta(y',V,E_Y) < (lex)\,\theta(y,V,E_Y)$. This is a contradiction. The nucleolus derived by E_Y must be contained in $IR(V) \cap \partial V(N)$. Then using (1) and the fact that $\mathcal{N}(V,e_Y)$ exists, (3) is obtained. ■

It is easy to observe that $\mathcal{N}(V, E_Y)$ satisfies symmetry. From (1) and (3), we see that $\mathcal{N}(V, E_Y)$ satisfies the core property. Indeed, $\mathcal{N}(V, e_Y)$ satisfies the core property.

Combining (2) and the fact that E_Y is continuous in x and V, we derive that $\mathcal{N}(V, E_Y)$ or $\mathcal{N}(V, e_Y)$ is not u.s.c. by Non-continuity Theorem.

4 An impossibility result for a solution

In this section, we will show that there is no solution for NTU games satisfying symmetry, continuity, and the core property.

Theorem 8 *There is no solution for NTU games with $|N| \geq 3$ satisfying non-emptiness, symmetry, continuity, and the core property.*

A class of 3-person games is constructed such that the required properties do not hold simultaneously.

Example 9 *Let σ be a solution satisfying non-emptiness, symmetry, continuity, and the core property.*

Consider the game $V \in \mathcal{G}_{\{1,2,3\}}$ given by

$$V(S) = G^S(\{(0,0,0)\}) \text{ for } S = \{1\}, \{2\}, \{3\}, \{1,2\}, \text{ and } \{1,3\},$$
$$V(S) = G^S\left(\left\{x \in \mathbb{R}_+^S \mid x(S) \leq 1\right\}\right) \text{ for } S = \{2,3\}, N.$$

The core of the game V is

$$C(V) = \{x \in V(N) \mid x(\{2,3\}) = 1 \text{ and } x_i \geq 0 \text{ for all } i \in N\}.$$

Let π be a permutation such that $\pi(1) = 3$, $\pi(2) = 2$, and $\pi(3) = 1$. Then π is a symmetry of V and $\sigma(V) = \left\{\left(0, \frac{1}{2}, \frac{1}{2}\right)\right\}$. Indeed, σ is non-empty, symmetric and satisfies the core property.

We are going to construct a sequence of games $\{V_k\}_{k=1}^{\infty}$ such that $V_k \to V$ as $k \to \infty$.

Let $E = \left(0, \frac{3}{4}, \frac{1}{4}\right)$, $A_k = \left(1 - \frac{1}{k+3}, 0, 0\right)$, $B_k = \left(0, 1 - \frac{1}{k+3}, 0\right)$, and $C_k = \left(0, 0, 1 - \frac{1}{k+3}\right)$, where k is any positive integer. Define the set $T_k = H(A_k, B_k, E) \cap H(A_k, C_k, E) \cap \mathbb{R}_+^N$, where $H(A_k, B_k, E)$ is the half space determined by the points A_k, B_k, and E and contains the origin 0. Similarly, the set $H(A_k, C_k, E)$ is defined. Hence, T_k is the convex hull determined by 0, A_k, B_k, C_k, and E. Please see Figure 2.

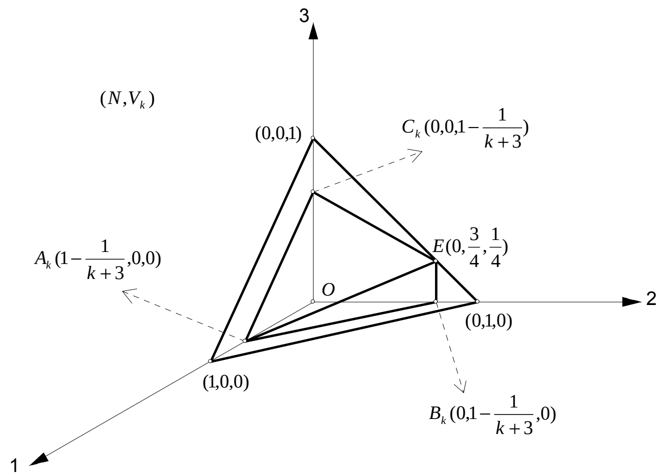

Figure 2.

Then we define $V_k \in \mathcal{G}_{\{1,2,3\}}$ as follows:

$$V_k(S) = V(S) \text{ for } S \in \mathcal{P} - \{N\} \text{ and } V_k(N) = G^{\{1,2,3\}}(T_k).$$

For every k, $\sigma(V_k) = C(V_k) = \left\{\left(0, \frac{3}{4}, \frac{1}{4}\right)\right\}$ by non-emptiness and the core property

We see that $V_k \to V$ as $k \to \infty$ in the topology induced by extended Hausdorff metric and $\sigma(V_k) \nrightarrow \sigma(V)$ as $k \to \infty$. It violates that σ is continuous.

5 The λ- excess function and the λ-nucleolus

The section consists of two subsections. In subsection 5.1, we define the λ-excess function $\tilde{e}$, and then, provide an axiomatization of it. In subsection 5.2, we first show that $\mathcal{N}(V,\tilde{e})$, the nucleolus of the NTU game V derived by $\tilde{e}$, is not u.s.c. Then the λ-nucleolus $\mathcal{N}_\lambda(V,\tilde{e})$ is defined by modifying the idea of the nucleolus slightly. In fact, $\mathcal{N}_\lambda(V,\tilde{e})$ satisfies criteria (A) and (B) to a large extend. More specific, $\mathcal{N}_\lambda(V,\tilde{e})$ satisfies the coincidence property, symmetry, u.s.c. It does not satisfy the core property but satisfies the inner core property given by Shapley and Shubik(1975) instead.

Besides $(A1)$ and $(A2)$, we require a game $V \in \mathcal{G}_N$ also satisfying:

$(A3)$ $V(N)$ is convex,
$(A4)$ there is a $x \in \mathbb{R}^N$ such that $V(S) \times \{O^{N \setminus S}\} \subseteq V(N) + x \;\forall\; S \in \mathcal{P}$,
$(A5)$ $\partial V(N)$ is smooth and the normal vector λ_x at $x \in \partial V(N)$ satisfies $(\lambda_x)_i > 0 \;\forall i \in N$.

$(A4)$ can be viewed as a weak monotonicity. We assume that λ_x is *normalized*, that is, $(\lambda_x)_1 = 1$ in the rest of the paper. It is interpreted to be "the comparison weights" for the utilities of players. Then $\lambda_x = e^N$ for $x \in \partial V(N)$ if V corresponds to a TU game.

We denote the class of games satisfying $(A1)$ - $(A5)$ to be $\mathcal{G}'_N$ and $\mathcal{G}'_N \subseteq \mathcal{G}_N$.

Let $y = (y_1, y_2, ..., y_n)$, $\omega = (\omega_1, \omega_2, ..., \omega_n) \in \mathbb{R}^N$, and $V \in \mathcal{G}'_N$. Denote $\omega \bullet y = \omega_1 y_1 + \omega_2 y_2 + \cdots + \omega_n y_n$ and $\omega * y = (\omega_1 y_1, \omega_2 y_2, ..., \omega_n y_n)$. If $\omega \in \mathbb{R}^N_{++}$, then $\omega^{-1} * y = \left(\frac{y_1}{\omega_1}, \frac{y_2}{\omega_2}, ..., \frac{y_n}{\omega_n}\right)$. Given $S \in \mathcal{P}$, let $(\omega * V)(S) = \{\omega * y : y \in V(S)\}$. Then $\omega * V \in \mathcal{G}'_N$.

Given $V \in \mathcal{G}'_N$ and $x \in \partial^+ V(N)$, we define $v_{\lambda_x} \in \mathbf{G}_N$ by

$$v_{\lambda_x}(S) = \max\{(\lambda_x)_S \bullet z : z \in V(S)\} \;\forall S \in \mathcal{P}. \tag{6}$$

$v_{\lambda_x}(S)$ is well-defined by the fact that $V(S)$ is a proper subset of $\mathbb{R}^S$, $(A1)$, $(A3)$ and $(A4)$. It is interpreted to be the worth of the coalition S with respect to λ_x.

Remark 10 *Let $\omega \in \mathbb{R}^N_{++}$. If $x \in \partial V(N)$, then $\omega * x \in \partial(\omega * V)(N)$. We see that $\lambda_x * \omega^{-1}$ is an normal vector of $\omega * V$ at $\omega * x \in \partial(\omega * V)(N)$. Hence, the normalized normal vector $\lambda_{\omega * x}$ at $\omega * x$ is $\omega_1 \cdot (\lambda_x * \omega^{-1})$.*

If we take $\omega = \lambda_x$, the normalized normal vector $\lambda_{\lambda_x * x}$ of $\lambda_x * V$ at $\lambda_x * x$ is $\lambda_{\lambda_x * x} = (\lambda_x)_1 \cdot (\lambda_x * \lambda_x^{-1}) = e^N$. We obtain that $(\lambda_x * V) + V_0$ corresponds to a TU game by $(A4)$.

$V \in \mathcal{G}'_N$ is called a *hyperplane game* if there are $h \in \mathbb{R}^N_{++}$ and a real constant c such that

$$V(N) = \{\, x \in \mathbb{R}^N : \; h \bullet x \leq c\}.$$

We call $v_0 \in \mathbf{G}_N$ the *zero game* if $v_0(S) = 0 \;\forall S \in \mathcal{P}$. Let V_0 correspond to v_0. Then $V_0 \in \mathcal{G}'_N$ and V_0 is called the zero game as well.

5.1 The λ- excess function

The λ-excess function for NTU games is proposed and it is the unique real-valued function satisfying the following three appealing properties: excess preservation, excess additivity, and excess invariance.

The λ-excess function $\tilde{e}$ maps from $\partial^+ V(N) \times \mathbb{R}^N \times \mathcal{P} \times \mathcal{G}'_N$ to $\mathbb{R}$ and

$$\tilde{e}(x, \mathbf{y}, S, V) = v_{\lambda_x}(S) - (\lambda_x * \mathbf{y})(S).$$

If $V \in \mathcal{G}'_N$ and it corresponds to $v \in \mathbf{G}_N$, then, for every $x \in \partial^+ V(N)$,

$$\tilde{e}(x, \mathbf{y}, S, V) = v_{\lambda_x}(S) - (\lambda_x * \mathbf{y})(S) = v(S) - \mathbf{y}(S) = e(\mathbf{y}, S, v).$$

That is, $\tilde{e}$ and e are the same on TU games and $\tilde{e}(0, \mathbf{0}, S, V_0) = e(\mathbf{0}, S, v_0) = 0$.

Remark 11 *Recall that both the excess for TU games e and an excess function for NTU games E are defined on the set of payoff vectors, coalitions, and games. Here one more factor $\partial^+ V(N)$ is added in the domain of the λ-excess function $\tilde{e}$. We will explain this at the end of the subsection.*

Let $\tilde{E}$ be a real-valued function defined on $\partial^+ V(N) \times \mathbb{R}^N \times \mathcal{P} \times \mathcal{G}'_N$.

Excess Preservation(EP): If $V \in \mathcal{G}'_N$ and it corresponds to $v \in \mathbf{G}_N$, then

$$\tilde{E}(x, \mathbf{y}, S, V) = e(\mathbf{y}, S, v) \ \forall x \in \partial^+ V(N) \ \ \forall \mathbf{y} \in \mathbb{R}^N \text{ and } \forall S \in \mathcal{P}.$$

Excess Additivity(EA): Given $V \in \mathcal{G}'_N$ and $x \in \partial^+ V(N)$, if $V + V_0 \in \mathcal{G}'_N$ and $x \in \partial^+(V + V_0)(N)$, then

$$\tilde{E}(x, \mathbf{y} + \mathbf{0}, S, V + V_0) = \tilde{E}(x, \mathbf{y}, S, V) + \tilde{E}(0, \mathbf{0}, S, V_0) \ \forall \mathbf{y} \in \mathbb{R}^N \text{ and } \forall S \in \mathcal{P}.$$

Excess Invariance(EI): Let $V \in \mathcal{G}'_N$ and $x \in \partial^+ V(N)$. If $\omega = (\omega_1, \omega_2, ..., \omega_n) \in R^N_{++}$, then

$$\tilde{E}(\omega * x, \omega * \mathbf{y}, S, \omega * V) = \omega_1 \cdot \tilde{E}(x, \mathbf{y}, S, V) \ \forall \mathbf{y} \in \mathbb{R}^N \text{ and } \forall S \in \mathcal{P}. \tag{7}$$

The interpretation of **EP** is that the value of the excess e for TU games is preserved, that is, the values of $\tilde{E}$ and e are the same on TU games. **EA** is that additivity of $\tilde{E}$ obtains if one is restricted to be zero game V_0 and the payoff vector 0. **EI** says that: If the payoffs in V are in utilities, the value of $\tilde{E}$ will not be affected if we employ different utility functions to represent the same real outcome. We will call ω_1 the *normalized factor* which will be explained later.

Lemma 12 *The λ- excess function $\tilde{e}$ satisfies **EP**, **EA**, and **EI**.*

Proof. We first show $\tilde{e}$ satisfying **EP.** Let V and v correspond to each other. If $x \in \partial^{+}V(N)$, $\mathbf{y} \in \mathbb{R}^{N}$ and $S \in \mathcal{P}$, then $\lambda_x = e^N$ and

$$\begin{aligned}
\tilde{e}(x, \mathbf{y}, S, V) &= v_{\lambda_x}(S) - (\lambda_x * \mathbf{y})(S) \\
&= \max\{(\lambda_x)_S \bullet z : z \in V(S)\} - (\lambda_x * \mathbf{y})(S) \\
&= \max\{(e^N)_S \bullet z : z \in V(S)\} - \mathbf{y}(S) \\
&= \max\{z(S) : z \in V(S)\} - \mathbf{y}(S) \\
&= e(\mathbf{y}, S, v).
\end{aligned}$$

Next, we show $\tilde{e}$ satisfying **EA.** Assume that $V + V_0 \in \mathcal{G}'_N$ and $x \in \partial^{+}(V + V_0)(N)$. Then $V + V_0$ is a TU game and $\lambda_x = e^N$. Given $\mathbf{y} \in \mathbb{R}^N$ and $S \in \mathcal{P}$, by **EP** and $\lambda_x = e^N$, we have

$$\begin{aligned}
\tilde{e}(x, \mathbf{y}, S, V + V_0) &= e(\mathbf{y}, S, V + V_0) \\
&= \max\{z(S) : z \in (V + V_0)(S)\} - \mathbf{y}(S) \\
&= \max\{z(S) : z \in V(S)\} - \mathbf{y}(S) \\
&= \max\{(\lambda_x)_S \bullet z : z \in V(S)\} - (\lambda_x * \mathbf{y})(S) \\
&= \tilde{e}(x, \mathbf{y}, V, S).
\end{aligned}$$

It remains to show that $\tilde{e}$ satisfies **EI.** Let $\omega = (\omega_1, \omega_2, ..., \omega_n) \in R^N_{++}$ and $x \in \partial^{+}V(N)$. Recall that $\lambda_{\alpha * x} = \alpha_1 \cdot (\lambda_x * \alpha^{-1})$ by Remark 10. Then

$$\begin{aligned}
& \tilde{e}(\omega * x, \omega * \mathbf{y}, S, \omega * V) \\
= \ & \max\{(\lambda_{\omega * x})_S \bullet z : z \in (\omega * V)(S)\} - (\lambda_{\omega * x} * (\omega * \mathbf{y}))(S) \\
= \ & \max\{(\omega_1 \cdot (\lambda_x * \omega^{-1}))_S \bullet (\omega * u) : u \in V(S)\} - (\omega_1 \cdot (\lambda_x * \omega^{-1})) * (\omega * \mathbf{y}))(S) \\
= \ & \omega_1 \cdot [\max\{(\lambda_x)_S \bullet u) : u \in V(S)\} - (\lambda_x * \mathbf{y})(S)] \\
= \ & \omega_1 \cdot \tilde{e}(x, \mathbf{y}, V, S).
\end{aligned} \tag{8}$$

■

Remark 13 *Given $V \in \mathcal{G}'_N$, $x \in IR(V) \cap \partial V(N)$, and $\omega \in \mathbb{R}^N_{++}$, $\omega * x \in IR(\omega * V) \cap \partial(\omega * V)(N)$. From Remark 10, we see that $(\lambda_x * \omega^{-1})$ is an normal vector of $\omega * V$ at $\omega * x$. So $c \cdot (\lambda_x * \omega^{-1})$ is an normal vector for every $c > 0$.*

We take the normalized normal vector to be the normal vector with the first component to be 1 and use it to calculate $\tilde{e}(\omega * x, \omega * \mathbf{y}, S, \omega * V)$. This is the reason why we have ω_1 in (8) and call it the normalized factor.

Lemma 14 *If $\tilde{E}$ satisfies **EP**, **EA**, and **EI**, then $\tilde{E} = \tilde{e}$.*

Proof. Let $x \in \partial^{+}V(N)$. Then $(\lambda_x * V) + V_0 \in \mathcal{G}'_N$ by Remark 10 and $(\lambda_x * x) + 0 \in \partial((\lambda_x * V) + V_0)(N)$.

Given $S \in \mathcal{P}$ and $\mathbf{y} \in \mathbb{R}^N$, by **EA, EP,** $e(\mathbf{0}, S, v_0) = 0$, **EI**, and the assumption that $(\lambda_x)_1 = 1$,

$$\begin{aligned} & \tilde{E}(\lambda_x * x, \lambda_x * \mathbf{y}, S, (\lambda_x * V) + V_0) \\ = \; & \tilde{E}(\lambda_x * x, \lambda_x * \mathbf{y}, S, \lambda_x * V) + \tilde{E}(0, \mathbf{0}, S, V_0) \\ = \; & \tilde{E}(\lambda_x * x, \lambda_x * \mathbf{y}, S, \lambda_x * V) + e(\mathbf{0}, S, v_0) \\ = \; & \tilde{E}(\lambda_x * x, \lambda_x * \mathbf{y}, S, \lambda_x * V) \\ = \; & (\lambda_x)_1 \cdot \tilde{E}(x, \mathbf{y}, S, V) \\ = \; & \tilde{E}(x, \mathbf{y}, S, V). \end{aligned} \tag{9}$$

Given $S \in \mathcal{P}$ and $\mathbf{y} \in \mathbb{R}^N$, by the fact that $(\lambda_x * V) + V_0$ is a TU game and **EP**, we have

$$\begin{aligned} & \tilde{E}(\lambda_x * x, \lambda_x * \mathbf{y}, S, (\lambda_x * V) + V_0) \\ = \; & e(\lambda_x * \mathbf{y}, S, (\lambda_x * V) + V_0) \\ = \; & \max\{z(S) : z \in ((\lambda_x * V) + V_0)(S)\} - (\lambda_x * \mathbf{y})(S) \\ = \; & \max\{z(S) : z \in (\lambda_x * V)(S)\} - (\lambda_x * \mathbf{y})(S) \\ = \; & \max\{(\lambda_x * u)(S) : u \in V(S)\} - (\lambda_x * \mathbf{y})(S) \\ = \; & \max\{(\lambda_x)_S \bullet u : u \in V(S)\} - (\lambda_x * \mathbf{y})(S) \\ = \; & \tilde{e}(x, \mathbf{y}, V, S). \end{aligned} \tag{10}$$

From (9) and (10), we derive the desired result. ■

Combining previous two lemmas, we have the following:

Theorem 15 *$\tilde{e}$ is the unique real-valued function defined on $\partial^{+}V(N) \times \mathbb{R}^N \times \mathcal{P} \times \mathcal{G}'_N$ satisfying **EP**, **EA**, and **EI**.*

The last part of the subsection is to explain the reason why one more factor $\partial^{+}V(N)$ is added in the domain of the λ-excess function $\tilde{e}$. If we rewrite excess preservation, excess additivity, and excess invariance in terms of the excess function E, there is no excess function satisfying these three properties.

Let E be an excess function for NTU games. That is, it is defined on $\mathbb{R}^N \times \mathcal{P} \times \mathcal{G}'_N$. We reformulate excess preservation, excess additivity, and excess invariance in terms of E in the following.

(α) If $V \in \mathcal{G}'_N$ and it corresponds to $v \in \mathbf{G}_N$, then

$$E(\mathbf{y}, S, V) = e(\mathbf{y}, S, v) \ \forall \mathbf{y} \in \mathbb{R}^N \text{ and } \forall S \in \mathcal{P}.$$

(β) If $V + V_0 \in \mathcal{G}'_N$, then

$$E(\mathbf{y} + \mathbf{0}, S, V + V_0) = E(\mathbf{y}, S, V) + E(\mathbf{0}, S, V_0) \ \forall \mathbf{y} \in \mathbb{R}^N \text{ and } \forall S \in \mathcal{P}.$$

(γ) Let $V \in \mathcal{G}'_N$. If $\omega = (\omega_1, \omega_2, ..., \omega_n) \in \mathbb{R}^N_{++}$, then

$$E(\mathbf{y}, S, V) = \omega_1 \cdot E(\omega * \mathbf{y}, S, \omega * V) \ \forall \mathbf{y} \in \mathbb{R}^N \text{ and } \forall S \in \mathcal{P}.$$

Remark 16 *Let $\omega = 2e^N$. Then $e(2e^N * \mathbf{y}, S, (2e^N * V) = 2 \cdot e(\mathbf{y}, S, V) \neq e(\mathbf{y}, S, V)$. Hence, it is obvious not to write (γ) to be*

$$E(\mathbf{y}, S, V) = E(\omega * \mathbf{y}, S, \omega * V) \ \forall \mathbf{y} \in \mathbb{R}^N \textit{ and } \forall S \in \mathcal{P}.$$

Theorem 17 *Let $\bar{\mathcal{G}}_N \subseteq \mathcal{G}'_N$. If E is a function defined on $\mathbb{R}^N \times \mathcal{P} \times \bar{\mathcal{G}}_N$ and satisfies (α), (β), and (γ), then $\bar{\mathcal{G}}_N$ consists of hyperplane games.*

Proof. Let $V \in \bar{\mathcal{G}}_N$ and $x \in \partial^+ V(N)$. We see that $\lambda_x * x \in \partial^+((\lambda_x * V) + V_0)(N)$ and $(\lambda_x * V) + V_0$ is a TU game by Remark 10.

For $S \in \mathcal{P}$ and an arbitrary $\mathbf{y} \in \mathbb{R}^N$, we have, by (γ), the facts that $(\lambda_x)_1 = 1$ and $E(\mathbf{0}, S, V_0) = e(\mathbf{0}, S, v_0) = 0$, (β), and (γ),

$$\begin{aligned}
E(\mathbf{y}, S, V) &= (\lambda_x)_1 \cdot E(\lambda_x * \mathbf{y}, S, \lambda_x * V) \\
&= E(\lambda_x * \mathbf{y}, S, \lambda_x * V) + E(\mathbf{0}, S, V_0) \\
&= E(\lambda_x * \mathbf{y}, S, (\lambda_x * V) + V_0) \\
&= e(\lambda_x * \mathbf{y}, S, (\lambda_x * V) + V_0) \\
&= \max\{(\lambda_x)_S \bullet z : z \in ((\lambda_x * V) + V_0)(S)\} - (\lambda_x * \mathbf{y})(S) \\
&= v_{\lambda_x}(S) - (\lambda_x * \mathbf{y})(S).
\end{aligned}$$

Since equality holds for every $x \in \partial^+ V(N)$ and arbitrary $\mathbf{y}$, λ_x must be a constant. We derive that V is a hyperplane game. ■

5.2 The λ-nucleolus

In this subsection, we will first show that $\mathcal{N}(V,\tilde{e})$ is not u.s.c. Then the λ-nucleolus $\mathcal{N}_\lambda(V,\tilde{e})$ is proposed by modifying the idea of the nucleolus slightly.

Using the same idea to define (4), $\mathcal{N}(V,\tilde{e})$ is defined as follows. Given $V\in\mathcal{G}'_N$, $x\in\partial^+V(N)$, and $y\in\mathbb{R}^N$, let

$$\theta(x,y,V,\tilde{e})) = (\tilde{e}(x,y,S_1,V),\cdots,\tilde{e}(x,y,S_{2^n-1},V)),$$

where $S_k\in\mathcal{P}$, $k=1,2,\cdots,2^n-1$, and $\tilde{e}(x,y,S_k,V)\geq\tilde{e}(x,y,S_{k+1},V)$ for $k=1,2,\cdots,2^n-2$. Then

$$\mathcal{N}(V,\tilde{e}) = \{\, x\in\partial^+V(N) : \theta(x,x,V,\tilde{e}) \leq (lex)\,\theta(x,y,V,\tilde{e}) \ \ \forall y\in IR(V)\}.$$

The following example shows that $\mathcal{N}(V,\tilde{e})$ is not u.s.c.

Example 18 *Consider the game $V\in\mathcal{G}'_{\{1,2,3\}}$ given by*

$$\begin{array}{l} V(\{i\}) = G^{\{i\}}(\{(0,0,0)\}) \textit{ for } i\in N, \\ V(S) = \{x\in\mathbb{R}^S : x(S)\leq 0\} \textit{ for } S=\{1,2\},\ \{1,3\}, \\ V(S) = \{x\in\mathbb{R}^S : x(S)\leq 1\} \textit{ for } S=\{2,3\},\ N. \end{array}$$

For every $x\in\partial^+V(N)$,

$$v_{\lambda_x}(S) = 0 \textit{ if } S\in\mathcal{P}-\{\{2,3\},N\},\ \textit{ and } v_{\lambda_x}(S)=1 \textit{ if } S=\{2,3\},N.$$

Then $\mathcal{N}(V,\tilde{e}) = \{(0,\frac{1}{2},\frac{1}{2})\}$.

Denote

$$\begin{aligned} A &= (1,0,0),\ B=(0,1,0),\ C=(0,0,1),\ A_k=(1-\frac{1}{k+3},0,0),\ \textit{and} \\ B_k &= (0,1-\frac{1}{k+3},0),\ \textit{where } k \textit{ is any positive integer.} \end{aligned}$$

Denote $H(A_k,B_k,C)$ to be a convex smooth surface containing A_k, B_k, and C such that $\partial V(N)=\{x\in\mathbb{R}^N : x(N)=1\}$ is the supporting hyperplane of $H(A_k,B_k,C)$ at C and $H(A_k,B_k,C)\to V(N)$ as $k\to\infty$ in the topology induced by the extended Hausdorff metric.

For every integer $k>0$, we define the game $V_k\in\mathcal{G}'_{\{1,2,3\}}$ to be

$$V_k(S) = V(S) \textit{ for } S\in\mathcal{P}-\{N\} \textit{ and } V_k(N) = H(A_k,B_k,C).$$

Then $V_k \to V$ as $k \to \infty$ in the topology induced by the extended Hausdorff metric.

For every positive integer k, $\tilde{e}(C, C, V_k, S) \leq 0\ \forall S \in \mathcal{P}$ and $\tilde{e}(C, x, V_k, \{2,3\}) > 0\ \forall x \in IR(V_k) - \{C\}$. Therefore, $C = \mathcal{N}(V_k, \tilde{e})$ for every k. Then $C = \mathcal{N}(V_k, \tilde{e}) \to C$ as $k \to \infty$. But $C \notin \mathcal{N}(V, \tilde{e})$. So $\mathcal{N}(V, \tilde{e})$ is not u.s.c.

Since V_k is not a hyperplane game for every $1 \leq k < \infty$ in the previous example, this inspires us to get hyperplane games involved if we would like to have an extension of the nucleolus satisfying u.s.c.

Given $V \in \mathcal{G}'_N$ and $x \in \partial^+ V(N)$, we define $V_x \in \mathcal{G}'_N$ to be

$$V_x(S) = \begin{cases} V(S), & \text{if } S \subsetneq N, \\ \{y \in \mathbb{R}^N \mid \lambda_x \bullet y \leq v_{\lambda_x}(S)\}, & \text{if } S = N. \end{cases}$$

Note that V_x is a hyperplane game and $IR(V) \subseteq IR(V_x)$. The λ-nucleolus $\mathcal{N}_\lambda(V, \tilde{e})$ of V is

$$\mathcal{N}_\lambda(V, \tilde{e}) = \{x \in \partial^+ V(N) : \theta(x, x, V, \tilde{e}) \leq (lex)\, \theta(x, y, V, \tilde{e})\ \ \forall y \in IR(V_x)\}.$$

In other words, instead of choosing the element to minimize the "total dissatisfaction" among $IR(V)$, $\mathcal{N}_\lambda(V, \tilde{e})$ is the set of feasible payoff vectors of V that minimize "total dissatisfaction" among $IR(V_x)$.

Lemma 19 *If $x \in \mathcal{N}_\lambda(V, \tilde{e})$, $\lambda_x * x$ is the nucleolus of the game v_{λ_x}, that is, $\{\lambda_x * x\} = \mathcal{N}(v_{\lambda_x}, e)$.*

Proof. Given $V \in \mathcal{G}'_N$ and $x \in \partial^+ V(N)$,

$$\begin{aligned} \lambda_x * V_x(N) &= \{\lambda_x * y \in \mathbb{R}^N : \lambda_x \bullet y \leq v_{\lambda_x}(N)\} \\ &= \left\{\lambda_x * y \in \mathbb{R}^N : (\lambda_x * y)(N) \leq v_{\lambda_x}(N)\right\} \\ &= \left\{z \in \mathbb{R}^N : z(N) \leq v_{\lambda_x}(N)\right\}. \end{aligned}$$

We see that $\partial^+(\lambda_x * V_x)(N) = I^*(v_{\lambda_x})$.

Given $y \in V_x(N)$ and $S \in \mathcal{P}$, we have

$$\tilde{e}(x, y, S, V) = v_{\lambda_x}(S) - (\lambda_x * y)(S) = v_{\lambda_x}(S) - z(S) = e(z, S, v_{\lambda_x}),$$

and hence,

$$\begin{aligned} \theta(x, y, V, \tilde{e})) &= (\tilde{e}(x, y, S_1, V), \cdots, \tilde{e}(x, y, S_{2^n-1}, V)) \\ &= (e(z, S_1, v_{\lambda_x}), \cdots, e(z, S_{2^n-1}, v_{\lambda_x})) = \theta(\lambda_x * y, v_{\lambda_x}, e), \end{aligned}$$

where $z = \lambda_x * y \in \lambda_x * V_x(N)$.

If $x \in \mathcal{N}_\lambda(V, \tilde{e})$, then $x \in \partial^+ V(N)$ and $\theta(x, x, V, \tilde{e}) \leq (lex)\, \theta(x, y, V, \tilde{e})$ $\forall y \in IR(V)$. We have that $\theta(\lambda_x * x, v_{\lambda_x}, e) \leq (lex)\, \theta(\lambda_x * y, v_{\lambda_x}, e)$ $\forall \lambda_x * y \in I^*(v_{\lambda_x})$. We derive that $\{\lambda_x * x\} = \mathcal{N}(v_{\lambda_x}, e)$. $\blacksquare$

Since $\lambda_x = e^N$ if V corresponds to a TU game v, we see that $\mathcal{N}_\lambda(V, \tilde{e}) = \mathcal{N}(v, e)$. That is, $\mathcal{N}_\lambda(V, \tilde{e})$ satisfies the coincidence property.

Theorem 20 *The λ-nucleolus is u.s.c. on $\mathcal{G}'_N$.*

Proof. Let $\{V_k\}_{k=1}^{\infty}$ be a sequence of games in $\mathcal{G}'_N$ and $V_k \to V$ as $k \to \infty$ in the topology induced by the extended Hausdorff metric and $x_k \to x \in \partial^+ V(N)$ as $k \to \infty$ in the topology induced by the Euclidean distance, where $x_k \in \mathcal{N}_\lambda(V_k, \tilde{e})$ $\forall\, k$.

We know that $\partial^+ V_k(N)$ is smooth for every k by $(A5)$. Hence, $\lambda_{x_k} \to \lambda_x$. We have $\lambda_{x_k} * x_k \to \lambda_x * x$ as $k \to \infty$ in the topology induced by the Euclidean distance.

Due to (3) and (6), we see $(v_k)_{\lambda_{x_k}} \to v_{\lambda_x}$ as $k \to \infty$ in the topology induced by the extended Hausdorff metric. Since $\{\lambda_x * x_k\} = \mathcal{N}\left((v_k)_{\lambda_{x_k}}, e\right)$ $\forall k$, $\lambda_{x_k} * x_k \to \lambda_x * x = \mathcal{N}(v_{\lambda_x}, e)$ as $k \to \infty$. Indeed, the nucleolus is continuous on TU games. We derive that $x \in \mathcal{N}_\lambda(V, \tilde{e})$. $\blacksquare$

The following example demonstrates that $\mathcal{N}_\lambda(V, \tilde{e})$ does not satisfy the core property.

Example 21 *Consider the game $V \in \mathcal{G}'_{\{1,2,3\}}$ given by*

$$\begin{array}{l} V(\{i\}) = G^{\{i\}}(\{(0,0,0)\}),\ \textit{where } i = 1,2,3, \\ V(S) = \{x \in \mathbb{R}^S : x(S) \leq 2\}\ \textit{for } S = \{1,2\}, \{2,3\}, \\ V(\{1,3\}) = G^{\{1,3\}}((1,0,2)), \\ V(N) = \left\{x \in \mathbb{R}^N \mid x_1 + x_2 + x_3 \leq 3\right\}. \end{array}$$

We see that $C(V) = [(1,1,1), (1,2,0)]$.

$\lambda = (1,1,1)$ is the normalized normal vector at every $x \in \partial^+ V(N)$. The game v_λ is

$$\begin{array}{l} v_\lambda(\{i\}) = 0,\ \textit{for } i = 1,2,3, \\ v_\lambda(\{1,2\}) = v_\lambda(\{2,3\}) = 2,\ \textit{and } v_\lambda(\{1,3\}) = v_\lambda(N) = 3. \end{array}$$

Then $\mathcal{N}(v_\lambda, e) = \left\{\left(\frac{4}{3}, \frac{1}{3}, \frac{4}{3}\right)\right\} \notin C(V)$.

The inner core given by Shapley and Shubik(1975) is defined as follows. Given $V \in \mathcal{G}'_N$, the inner core of V is

$$\begin{aligned} IC(V) &= \bigcup_{x \in \partial^+ V(N)} \{\mathbf{y} \in IR(V) : (\lambda_x * \mathbf{y})(S) \geq v_{\lambda_x}(S) \ \forall S \in \mathcal{P}\} \quad (11) \\ &= \bigcup_{x \in \partial^+ V(N)} \{\mathbf{y} \in IR(V) : \tilde{e}(x, \mathbf{y}, V, S) \leq 0 \ \forall S \in \mathcal{P}\}. \end{aligned}$$

Let σ be a solution for NTU games. We say that σ satisfies *the inner core property* if $IC(V) \neq \emptyset$ and $\sigma(V) \subseteq IC(V)$.

Theorem 22 *The λ-nucleolus for NTU games satisfies the inner core property and symmetry.*

These results can be derived straightforwardly. We omit the proofs.

Remark 23 *For TU games, property I and property II are interesting because they are able to characterize the nucleolus and are helpful to prove continuity of the nucleolus. For details, please see Kohlberg*(1971)*. From Lemma* 19*, we see that the λ-nucleolus also shares these two properties in the following sense.*

Given $V \in \mathcal{G}'_N$, $x \in \mathcal{N}_\lambda(V, \tilde{e})$ if and only if the coalition array $b_1, \cdots, b_p$ that belongs to $(\lambda_x * x, v_{\lambda x})$ satisfies property I and property II.

6 Shapley procedure

Shapley(1969) proposed a procedure to extend the Shapley value to the NTU value for NTU games. The section is to illustrate that the procedure might be a nice way to take if one tries to extend a solution for TU games to NTU cases.

Given a solution for TU games φ, we say that the solution $\bar{\varphi}$ for NTU games is derived from φ via Shapley procedure if, given a game V, $\bar{\varphi}(V)$ consist of payoff vectors x such that $x \in \partial^+ V(N)$ and

$$\lambda_x * x \in \varphi(v_{\lambda_x}). \quad (12)$$

More specific, $\bar{\varphi}$ is the NTU value if φ is the Shapley value and $\bar{\varphi}$ is the inner core if φ is the core. Although the λ-nucleolus is derived by a different way, it turns out to be the nucleolus version derived by the procedure by Lemma 19.

This relationship (12) is crucial to prove that the λ-nucleolus is u.s.c. Aumann(1985) also took advantage of it to axiomatize the NTU value. Since (12) provides a nice connection between φ and $\bar{\varphi}$, we can easily extend these two results to other solutions for NTU games derived via the procedure.

Besides the basic axioms given by Aumann(1985) , we add *the coincidence property* to axiomatize $\bar{\varphi}$. The coincidence property plays a bridge between a solution for TU games and its corresponding solution for NTU games.

Given a solution for TU games φ, we denote $\mathbf{G}_{\varphi} \subseteq \mathbf{G}_N$ to be the set of games with $\varphi(v) \neq \emptyset \; \forall v \in \mathbf{G}_{\varphi}$ and denote $\mathcal{G}_{\varphi} \subseteq \mathcal{G}'_N$ to be the set of games with $\bar{\varphi}(V) \neq \emptyset \; \forall V \in \mathcal{G}_{\varphi}$.

Theorem 24 *Let φ be a solution defined on $\mathbf{G}_{\varphi}$. φ is continuous if it is single-valued or is u.s.c. if it is set-valued. Then $\bar{\varphi}$ is u.s.c. on $\mathcal{G}_{\varphi}$.*

The result can be obtained by using the same idea as what we have done on Theorem 20. We omit the proof.

Let φ be a solution defined on $\mathbf{G}_{\varphi}$ satisfying the following properties:

(α) $\varphi(v_0) = 0$, where $v_0 \in \mathbf{G}_{\varphi}$ is the zero game;

(β) Given u, $v \in \mathbf{G}_{\varphi}$, if u and v are S-equivalent, that is, there exists a $r > 0$ and $\alpha_1, \alpha_2, ..., \alpha_n$ are in $\mathbb{R}$ such that $v(S) = ru(S) + \alpha(S)$ for all $S \subseteq N$, then $\varphi(v) = r\varphi(u) + (\alpha_1, \alpha_2, ..., \alpha_n)$; and

(γ) $\varphi(v) \subseteq I(v)$.

There are many well-known solutions for TU games satisfying (α) - (γ), for instance, the Shapley value, the nucleolus, etc.

Let σ be a solution satisfying the following axioms for all games U, V, $W \in \mathcal{G}_{\varphi}$:

Axiom 1 Non-emptiness(NE) : $\sigma(V) \neq \emptyset$.

Axiom 2 Efficiency(EFF): $\sigma(V) \subseteq \partial V(N)$.

Axiom 3 Restricted Additivity(RA) : If $U = V + V_0$, then

$$\sigma(U) \supset (\sigma(V) + \sigma(V_0)) \cap \partial U(N) .$$

Axiom 4 Scale Covariance(SC) : $\sigma(\lambda * V) = \lambda * \sigma(V)$.

Axiom 5 Independence of Irrelevant Alternatives : If $V(N) \subseteq W(N)$ and $V(S) = W(S)$ for all $S \in \mathcal{P} - \{N\}$, then $\sigma(V) \supseteq \sigma(W) \cap V(N)$.

Axiom 6 Coincidence Property(CP) : If $V \in \mathcal{G}_{\varphi}$ and $v \in \mathbf{G}_N$ correspond to each other, then $\sigma(V) = \varphi(v)$.

For interpretations of *Axioms* 1, 2, 4, and 5, please see Aumann(1985). Restricted additivity means that if x is in the solution $\sigma(V)$ and it does happen to be efficient in $V+V_0$, then it is also in the solution $\sigma(V+V_0)=\sigma(U)$. That is, additivity obtains if one is restricted to be the zero game. If one claims to extend a solution for TU games to NTU cases, it is reasonable to require that it coincides with its original version on TU games. This is what *Axiom* 6 interprets.

Remark 25 *The axiomatic system given here is slightly different from the one by Aumann. The differences are:*

(1) *Unanimity which is essential for the NTU value is replaced by Coincidence Property. Otten and Peters*(2002) *used it too.*

(2) *Conditional Additivity is replaced by Restricted Additivity.*

(3) *Closure Invariance is deleted because all games considered in the paper are closed.*

Lemma 26 $\bar{\varphi}$ *satisfies* NE, EFF, RA, SC, IIA,*and* CP *on* $\mathcal{G}_\varphi$.

To prove the Lemma, we follow the idea of the proof of Lemma 7.3 of Aumann(1985).

Proof. $\bar{\varphi}$ satisfies NE because it is defined on $\mathcal{G}_\varphi$. $\bar{\varphi}$ satisfies EFF and CP because of (12). IIA and SC can be obtained straightforwardly. Note that we need (β) to have SC.

To show RA, let $y\in\bar{\varphi}(V)$, $0\in\bar{\varphi}(V_0)$, and $y+0=y\in\partial U$; we wish to show $y\in\bar{\varphi}(U)$. Let λ_y be the normalized normal vector of U at y. Then $\lambda_y=(1,1,...,1)$ because U corresponds to a TU game. Since $v+v_0=v$ and $\varphi(v_0)=0$ by (α), we have $\varphi(v+v_0)=\varphi(v)+\varphi(v_0)$. Then RA can be derived by following what Aumann showed the NTU value satisfies Conditional Additivity. ■

Lemma 27 $\bar{\varphi}(V)=\sigma(V)\ \ \forall V\in\mathcal{G}_\varphi$

The Lemma can be derived by following the proof of Lemmas 8.6 and 8.7 of Aumann(1985) step by step. We omit the proof. Note what Lemma 8.1 of Aumann states is Coincidence Property. Then we obtain the following result from previous two lemmas.

Theorem 28 $\bar{\varphi}$ *is the unique solution satisfying* NE, EFF, RA, SC, IIA,*and* CP.

7 Final words

(I) Four properties of three extensions of the nucleolus are scrutinized. We summarize the results in the following:

	$\mathcal{N}(V, E_K)$	$\mathcal{N}(V, e_Y)$	$\mathcal{N}_\lambda(V, \tilde{e})$
the coincidence property	$-$	$+$	$+$
symmetry	$-$	$+$	$+$
the core property	$+$	$+$	$-^*$
usc	$-$	$-$	$+$

*The λ-nucleolus satisfies the inner core property.

(II) As for existence issue of $\bar{\varphi}$, please see Shapley(1953) and Otten and Peters(2002) . Qin(1994) studied existence of the inner core.

(III) For $V \in \mathcal{G}'_N$, we require that $\partial V(N)$ is smooth and show that the λ- nucleolus is u.s.c. Otten and Peters(2002) considered the class of games in which $\partial V(N)$ is not necessarily smooth. The following example illustrates that the λ- nucleolus is not u.s.c in their setting.

Example 29 *Let $O = (0,0)$, $A = (1,0)$, $B = (0,1)$, and the curve $\Gamma = \{(x,y) \in \mathbb{R}^N_+ : y = 1 - x^2\})$.*

Denote $M = (\frac{1}{2}, \frac{1}{2})$ and $C = (\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2})$ to be the midpoint of $\overline{AB}$ and the intersection point of the ray $\overrightarrow{OM}$ and Γ, respectively.

Let $N = \{1,2\}$. Consider (N,V) and V is given by

$$\begin{array}{l} V(\{i\}) = \{0\} \text{ for } i = 1,2, \text{ and} \\ V(N) = G^N(\Gamma) \cap \mathbb{R}^N_+ = \{(x,y) \in \mathbb{R}^N_+ : x^2 + y \leq 1\}). \end{array}$$

The line $y - \frac{\sqrt{5}-1}{2} = -2(\frac{\sqrt{5}-1}{2})(x - \frac{\sqrt{5}-1}{2})$ tangents to $\Gamma = \partial^+ V(N)$ at C. The normalized normal vector λ_C of Γ at C is $(1, \sqrt{5}-1)$. Then

$$v_{\lambda_C}(\{i\}) = 0 \text{ for } i = 1,2, \text{ and } v_{\lambda_C}(N) = \frac{5-\sqrt{5}}{2}.$$

Clearly, $\mathcal{N}(v_{\lambda_C}, e) = \{(\frac{5-\sqrt{5}}{4}, \frac{5-\sqrt{5}}{4})\}$. $C \notin \mathcal{N}_\lambda(V, \tilde{e})$. Indeed,

$$\begin{aligned} \lambda_C * C &= (1, \sqrt{5}-1) * (\frac{\sqrt{5}-1}{2}, \frac{\sqrt{5}-1}{2}) = (\frac{\sqrt{5}-1}{2}, (\sqrt{5}-1)(\frac{\sqrt{5}-1}{2})) \\ &\neq (\frac{5-\sqrt{5}}{4}, \frac{5-\sqrt{5}}{4}). \end{aligned}$$

Let $\Gamma_1 = \{(x,y) \in \mathbb{R}^N_+ : x^2 + y = 1$ *and* $0 \leq x \leq \frac{\sqrt{5}-1}{2}\}$ *and* $\Gamma_2 = \{(x,y) \in \mathbb{R}^N_+ : x^2 + y = 1$ *and* $\frac{\sqrt{5}-1}{2} \leq x \leq 1\}$. *Hence* $\Gamma = \Gamma_1 \cup \Gamma_2$ *and* $\Gamma_1 \cap \Gamma_2 = \{C\}$.

For each $k \geq 0$, *we denote*
(i) $M_k = M + (\frac{2^k-1}{2^k})(C - M) = (\frac{1}{2^k})M + (\frac{2^k-1}{2^k})C$,
(ii) L_k *is the line parallel to* $\overline{AB}$ *passes through* M_k,
(iii) B_k *is the intersection point of the line* L_k *and the curve* Γ_1, *and*
(iv) A_k *is the intersection point of the line* L_k *and curve* Γ_2. *Please see Figure 3.*

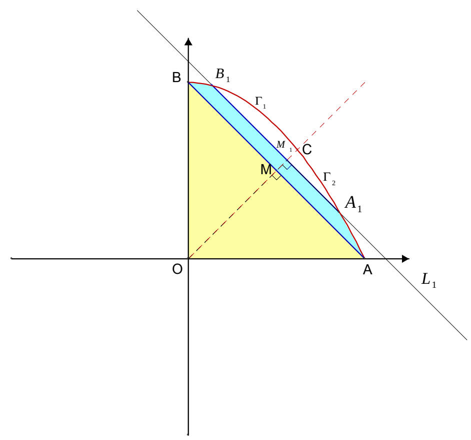

Figure 3.

Define (N, V_k) *as follows. For each* $k \geq 1$,

$$V_k(\{i\}) = V(\{i\}) \text{ for } i = 1,2, \text{ and}$$
$$V_k(N) = \text{the region enclosed by } x\text{-axis, } y\text{-axis, } L_k, \text{ and } \Gamma.$$

We see that $\partial\left(V_k\left(N\right)\right)$ *is not smooth for every* k. *But* $V_k \to V$ *in the topology induced by the Hausdorff metric.*

Let $\lambda = (1,1)$. *For each* $k \geq 1$, *we define* $(N, (v_k)_\lambda)$ *to be:*

$$(v_k)_\lambda(\{i\}) = \{0\} \text{ for } i = 1,2, \text{ and}$$
$$(v_k)_\lambda(N) = M_k\left(N\right) = \left(\tfrac{1}{2^k}M + \tfrac{2^k-1}{2^k}C\right)\left(N\right).$$

One can easily check that $\lambda * M_k = M_k \in \mathcal{N}(v_{\lambda_{M_k}}, e)$ *for each* $k \geq 1$, *and hence,* $M_k \in \mathcal{N}_\lambda\left(V_k, \tilde{e}\right)$ *for each* $k \geq 1$. *It implies that* $M_k \to C \notin \mathcal{N}_\lambda\left(V, \tilde{e}\right)$ *as* $k \to \infty$. *So* $\mathcal{N}_\lambda\left(V, \tilde{e}\right)$ *is not u.s.c.*

References

[1] Aliprantis CD and Borders KC(2006) Infinite Dimensional Analysis: A Hitchhiker's Guide. Springer.

[2] Aumann R(1985) An axiomatization of the non-transferable utility value. Econometrica 53: 599-612.

[3] Dugundji J(1966) Topology. Allyn and Bacon.

[4] Kalai E (1975) Excess function for cooperative game without side payments. SIAM J Appl Math 29: 60-71.

[5] Kohlberg E (1971) On the nucleolus of a characteristic function game. SIAM J Appl Math 20: 62-66.

[6] Kaneko M and Wooders M(2004) Utility theories in cooperative games. Handbook of Utility Theory II Eds. S. Barbera et al. Kluwer Academic Pub.

[7] Nakayama M (1983) A note on a generalization of the nucleolus to games without side payments. Int J Game Theory 12: 115-122.

[8] Otten GJ and Peters H(2002) The Shapley transfer procedure for NTU games. Chapters in game theory: in honor of Stef Tijs, Kluwer Academic Pub.

[9] Qin C.-Z. (1994) The Inner Core of an n-Person Game. Games and Economic Behavior 6: 431-444.

[10] Schmeidler DN (1969) The nucleolus of a characteristic function game form. SIAM J Appl Math 17: 1163-1170.

[11] Shapley LS (1969) Utility comparison and the theory of games. In La Decision, Paris: Editions du CNRS, 251-263.

[12] Shapley LS and Shubik M(1975) Competitive outcomes in the cores of market games. Int J Game Theory 4: 229-237.