On Efficient Private-Information Allocation with Capital*

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Abstract

This paper studies the efficient allocation in an informationally constrained economy with production. It reformulates the Atkeson and Lucas (1992) model by introducing a constant return to scale production technology. I show the existence of an efficient, incentive compatible allocation which strictly improves the welfare of agents from the autarkic case. In addition, I discuss how the result of this paper and the rationale behind it can be applied to related literature as Atkeson and Lucas (1992) and Khan and Ravikumar (2001).

1 Introduction

Atkeson and Lucas (1992) study the dynamics of agents' wealth distribution in an exchange economy with private information. There is a continuum of agents living in the economy. Agents face a privately observed idiosyncratic preference shock, which is identically and independently distributed over periods and across agents. In each period, agents share a constant amount of consumption goods by reporting their preference type (history) [1]. Atkeson and Lucas show the existence of an efficient incentive compatible allocation and propose a recursive method to characterize it as well. However, the pure exchange economy environment of the model is not satisfactory enough in the following respects: firstly, macroeconomic study typically involves capital and production in a model, therefore a natural research question to raise is that whether or not there exists such an efficient incentive compatible allocation in an economy with capital. Furthermore, the

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[1] It is equivalent to assume that the agent only makes the preference report of current period, and the planner can perfectly recall the reporting history of each agent. In later part of this paper, this assumption is used.

existence of capital may weaken the mutual insurance role that the efficient allocation plays. Then if there does exist an efficient incentive compatible allocation, whether or not it can (partially) insure agents against their idiosyncratic preference shocks?

This paper reformulates Atkeson and Lucas (1992) model with capital: assume that the economy is endowed with some goods only in the initial period and nothing later on. Endowment can be used either as consumption or as capital, which produces goods with a constant return to scale production technology. Similar as in Atkeson and Lucas (1992), there is also a continuum of agents living in the economy and agents face a privately observable i.i.d preference shock in each period.

Pareto optimality implies that consumption of agents in any given period only depends on their current reporting preference types; the higher the marginal utility of current consumption, the higher the contemporaneous consumption level. However, it violates agents' incentive compatibility. Since preference shocks are agents' private information, the incentive compatibility problem can be solved by conditioning agents' current consumption not only on their current reports but also on their reporting histories. Consider that agents are virtually assigned with the same amount of capital goods initially, which is addressed as the claim of capital in this paper. Assume that the claim of capital is publicly observable, then it serves as not only an incentive device but also a recording device. In any period, an agent makes a report about his/her current preference types, then the consumption today is assigned along with the claim of capital tomorrow. Agents are induced to report truthfully in the following way: given the same claim of capital today, agents with a higher instantaneous marginal utility value are assigned with more consumption goods today and a lower claim of capital tomorrow. On the other hand, agents with a lower instantaneous marginal utility value today are assigned with fewer consumption goods today but a higher claim of capital tomorrow. Therefore, not only the claim of capital can be used as an incentive device to motivate agents to truthfully report their current preference types, but also it summarizes the reporting histories of agents. Since agents are identical ex ante and preference shocks are i.i.d over agents, it is enough to consider the representative agent. The aggregate feasibility constraint in the initial period is satisfied due to the law of large numbers. Moreover, the constant return to scale production technology and the feasibility of the investment plan imply that the feasibility constraint are satisfied in each period. In contrast, the recursive problem of the pure exchange economy model requires tracking agents' wealth distribution over the time. Besides, the bellman equation in this paper is different from those in previous literature. Instead of minimizing the initial resource level to attain some ex ante expected utility level, I solve the utility maximization problem directly.

Analysis shows that there exists a unique efficient, incentive compatible allocation in this economy. Furthermore, this allocation can strictly improve agents' ex ante expected discounted utility from the autarkic case. The pareto improvement brought by the optimal allocation is due to the improvement in risk sharing among agents rather than the economies of scale.

The last part of the paper discusses how this paper fits in dynamic contract literature[2]. In particular, how this paper relates with Atkeson and Lucas (1992) and Khan and Ravikumar (2001): first, this paper reformulates the model in Atkeson and Lucas (1992) by introducing a constant return to scale production technology[3]. As a consequence, the period resource constraints are endogenously determined by the capital level. The higher the rate of return, the higher the consumption level today as well as the higher the claim of capital tomorrow. Moreover, no aggregate uncertainty implies that there exists some rate of return R such that the capital level is constant over the period, thus the aggregate consumption. In this case, the efficient, incentive compatible allocation in an economy with capital is feasible in the exchange economy whose endowment level coincides with the aggregate consumption in the economy with capital. Since the feasible allocation set of the latter economy is a proper subset of the feasible allocation set of the previous one, if a feasible allocation in the latter economy is efficient in the previous one, then it must be efficient in the latter one. Therefore, the existence of an information constrained allocation in an endowment economy can be shown as a corollary of the results in this paper. Secondly, the economic environment in this paper is similar to the one in Khan and Ravikumar (2001) except for the type of shocks that agents face. In Khan and Ravikumar (2001), agents experience a privately observed income shock and trade with a competitive financial intermediary. The equilibrium allocation of the model is studied. Nevertheless, it can be shown that the informationally constrained efficient allocation analyzed in this paper can be viewed as a competitive equilibrium allocation as well. Even more, the same logic of proof can be used to verify the existence of a unique informationally constrained efficient allocation in Khan and Ravikumar (2001). In addition, there exists a one-to-one mapping from an economy with income shocks to an economy with preference shocks such that capital grows at the same rate in these 2 economies under the informationally constrained efficient allocations. Thus as in the economy with income shocks, incomplete risk sharing of preference shocks reduces the capital growth rate compared to the full risk

[2]Related literature include Spear and Srivastava (1987), Green (1987), Thomas and Worrall (1990), Taub (1990), Phelan and Townsend (1991), and Marcet and Marimon (1992). Different from these earlier papers, Atkeson and Lucas (1992) study the planner's problem which involves a period-by-period constant resource constraint. Moreover, they show that the one-to-one principle agent problem studied by the earlier literature can be viewed as a decentralized version of the model as they study.

[3]There is another line of research which studies the incentive compatibility in an economic environment with capital started with Kehoe and Levine (1993), followed by Kocherlakota (1996) and Alvarez and Jermann (2000). They tend to explain the existing incompleteness of the social insurance by introducing the incentive compatibility problem due to limited commitment. It is different from that caused by private information which is studied in the model here.

sharing case[4].

2 Economic Environment

This section offers a formal description of the model. It is a discrete, infinite horizon model.

2.1 Social Endowment and Production

The economy is endowed with k_0 units of composite goods in the initial period. Goods can be used either as consumption or as capital which can be used to produce goods later at a constant gross rate of return, $R > 1$.

In each period, returns from the production are divided into 2 parts: current consumption and later capital. The law of motion of the capital can be expressed by $c_t + k_{t+1} = Rk_t$.

2.2 Agents

There is a continuum of ex ante identical agents. Agents live for infinite periods. Each agent faces an idiosyncratic serially independent preference shock in each period. Preference shocks are privately observable.

Formally, let A be the space of agents and μ be a non-atomic Borel measure on A. Without loss of generality, assume that $\mu(A) = 1$.

Let Ω be a sample space and M be the space of preference shocks, $M = \{\theta^1, \theta^2, \ldots, \theta^N\}$, where $\theta^1 < \theta^2 < \ldots, < \theta^N$. In addition, let m be the distribution of θ^i on M.

For simplicity, assume that

Assumption 2.1 $\forall t \in N$, $\forall a \in A$, $E[\Theta_t^a(\omega)] = 1$.

The preference shock of agent a in period t is a random variable, $\Theta_t^a : \Omega \to M$, and denote the realization of $\Theta_t^a(\cdot)$ as θ_t^a.

Let $< \vec{\theta} >_t$ be the $t-$period preference type history, $< \vec{\theta} >_t = \{\theta_1, \ldots, \theta_t\}$, (when $t = 0$, $< \vec{\theta} >_0$ is a null sequence), $M^t = \{< \vec{\theta} >_t\}$ be the t-fold product space, and m^t be the product measure on M^t.

[4]See Khan and Ravikumar (2001) proposition 2 and the results of numerical simulations.

Accordingly, let $\vec{\theta}$ be the complete preference history, $\vec{\theta} = \{\theta_1, \theta_2, \ldots, \}$, M^∞ be the space of $\vec{\theta}$, and the m^∞ be measure on M^∞.

In addition, let $(\vec{\theta}, t)$ be a node of the preference history, which is defined by a $t-$period preference history $< \vec{\theta} >_t$. In turn, $(\vec{\theta}, t)$ induces N preference histories of $(t+1)-$period $\sigma^i_{\vec{\theta},t}$ as follows,

$$\sigma^i_{\vec{\theta},t} = \{\theta_1, \ldots, \theta_t, \theta^i\},$$

where $\theta^i \in \{\theta^1, \ldots, \theta^N\}$.

Each agent has the preference over the consumption streams, which is ordered by

$$E[\sum_{t\in\mathbb{N}} \delta^{t-1}\Theta_t(\omega)u(c_t)]. \tag{1}$$

$\delta \in [0, 1]$ is the discount factor and the period utility function $u(\cdot)$ satisfies the following assumption:

Assumption 2.2 $u(\cdot) : (0, \infty) \to \mathcal{R}$ *is an increasing, strictly concave, twice continuously differentiable function, and* $\lim_{c\to\infty} u'(c) = 0$.

2.3 Reporting Strategy

In each period, each agent has an opportunity to report a message about his/her current preference type. Given the assumption that preference shocks are agents' private information, they may make counterfactual reports.

Formally, denote a reporting strategy of an agent as $\vec{z} = \{z_t\}_{t=1}^\infty$, where $z_t : M^t \to M$, satisfies that for all $\vec{\theta}$, $\vec{\theta'} \in M^\infty$, if

$$< \vec{\theta} >_t = < \vec{\theta'} >_t,$$

then

$$z_t(< \vec{\theta} >_t) = z_t(< \vec{\theta'} >_t).$$

Denote the space of reporting strategies as Z.

Definition 2.3 *The truth-telling reporting strategy* $\vec{z}^*$ *satisfies that for any* $t \geq 1$, $\vec{\theta} \in M^\infty$, $z^*_t(< \vec{\theta} >_t) = \theta_t$.

The rest of the paper is organized as follows: in section 3, the planner's maximization problem is formulated in a recursive way and a unique informationally constrained efficient allocation is shown to exist. Section 4 extends the result of this paper to the exchange economy described by Atkeson and Lucas (1992). It shows that the existence of the informationally constrained efficient allocation in an exchange economy can be verified as a corollary of the existence results in this paper. Section 5 discusses the relationship between the preference shock discussed in this paper and the income shock studied in Khan and Ravikumar (2001). The proofs of this paper are collected in the appendix.

3 The Informationally Constrained Efficient Allocation

This section studies the existence of efficient allocation in the informationally constrained economy with capital. The incentive compatibility problem of agents is resolved by conditioning their contemporaneous consumption not only on current preference types but their reporting history as well. Moreover, I study the planner's problem in its recursive form as the space of reporting histories grows exponentially as time goes by. A bellman equation is constructed accordingly. The solution to the Bellman equation characterizes the efficient allocation subject to the incentive compatibility if it does exist. To be noted that the Bellman equation constructed in this model is different from the one in Atkeson and Lucas (1992) in 2 aspects. First, it directly maximizes agents' expected discounted utility instead of solving its dual problem as discussed in previous literature. In addition, capital serves as the one dimension state variable of the Bellman equation compared to the infinite dimension one used in Atkeson and Lucas (1992)[5].

3.1 Allocation, Incentive Compatibility and Temporary Incentive Compatibility

To solve the incentive compatibility problem, an allocation[6] $\vec{\Gamma}$ assigns the consumption of agents conditioning on not only their current preference report but also their histories of reports as well. Therefore, an allocation $\vec{\Gamma} = \{\Gamma(\vec{\theta}, t)\}_{t=1}^{\infty}$ is composed of a sequence of functions, $\Gamma(\vec{\theta}, t) : M^t \to R_+$.

Define $U(\vec{\Gamma}, \vec{\theta}, t)$ to be the value function of an agent's expected utility discounted to period t, of consumption from period t and afterwards,

$$U(\vec{\Gamma}, \vec{\theta}, t) = E_{\vec{\theta}}[\sum_{\tau=t+1}^{\infty} \delta^{\tau-1}\theta_t u(\Gamma(\vec{\theta}, \tau))| < \vec{\theta} >_t], \quad (2)$$

[5]Without capital, Atkeson and Lucas (1992) have to use agents' promised utility distribution as a state variable of the Bellman equation.

[6]Since all agents are identical ex ante, in this model, only anonymous allocations are considered.

where $E_{\vec{\theta}}[\cdot| < \vec{\theta} >_t]$ is the expectation taken with respect to $\vec{\theta}$ conditional on the preference history $< \vec{\theta} >_t$. In particular, when $t = 0$, $U(\vec{\Gamma}, \vec{\theta}, 0)$ denotes the ex ante expected discounted utility of an agent given the allocation $\vec{\Gamma}$.

The fact that preference shocks of agents are their private information implies that an agent's actual consumption stream depends on his/her reporting strategy, $\vec{\Gamma} \circ \vec{z} = \{\Gamma(\vec{z}(\vec{\theta}), t)\}_{t=1}^{\infty}$. In turn, the ex ante expected discounted utility of an agent is $U(\vec{\Gamma} \circ \vec{z}, \vec{\theta}, 0)$.

An allocation is incentive compatible if for any $\vec{\theta} \in M^{\infty}$ it is optimal for an agent to report truthfully[7]. That is, for any $\vec{\theta} \in M^{\infty}$, and all $\vec{z} \in Z$,

$$U(\vec{\Gamma} \circ \vec{z}^{*}, \vec{\theta}, 0) = U(\vec{\Gamma}, \vec{\theta}, 0) \geq U(\vec{\Gamma} \circ \vec{z}, \theta, 0). \tag{3}$$

In addition, follow the notion as in Green (1987), an allocation is temporarily incentive compatible at a node $(\vec{\theta}, t)$, if given the preference history $< \vec{\theta} >_{t-1}$, it is optimal for an agent to report truthfully in current period,

$$\theta^i u(\Gamma(\vec{\sigma}^i_{\vec{\theta},t-1}, t)) + \delta U(\vec{\Gamma}, \vec{\sigma}^i_{\vec{\theta},t-1}, t) \geq \theta^i u(\Gamma(\vec{\sigma}^j_{\vec{\theta},t-1}, t)) + \delta U(\vec{\Gamma}, \vec{\sigma}^j_{\vec{\theta},t-1}, t), \tag{4}$$

for all $\theta^i, \theta^j \in M$ and $\theta^i \neq \theta^j$.

An allocation is temporarily incentive compatible if for any $\theta \in M$, $t > 0$, it is optimal for an agent to report truthfully at node $(\vec{\theta}, t)$.

Consider allocations that satisfy incentive compatibility (3) and the transversality condition [8] as follows

$$\lim_{t \to \infty} \sup_{z \in Z} |\delta^t U(\vec{\Gamma} \circ z, \vec{\theta}, t)| = 0 \tag{5}$$

Denote the space of allocations that satisfy the above conditions as S.

The following lemma states that a temporarily incentive compatible allocation is incentive compatible if it satisfies the transversality condition (5).

Lemma 3.1 *If an allocation $\vec{\Gamma}$ satisfies the transversality condition (5), then $\vec{\Gamma}$ is incentive compatible if and only if it is temporarily incentive compatible.*

[7] In this paper, only the truth telling Nash equilibrium is considered. Thus the allocation is defined by assuming that all other agents are telling the truth.

[8] According to the idealized law of large numbers, allocations that violate the transversality condition assign a positive fraction of agents with infinitely large consumptions. This implies that these allocations require a infinitely high initial endowment, which are infeasible for any given endowment level k^0. Without loss of generality, analysis can be focused on allocations which satisfy the transversality conditions only.

Note that the reporting strategies studied in this model share a similar structure as those in Green (1987), thus lemma (3.1) can be shown using the same logic as lemma 2 in Green (1987). Find the formal proof in the appendix.

Moreover, according to the idealized law of large numbers, an allocation $\vec{\Gamma}$ is feasible if it satisfies

$$E_\theta[\sum_{t=1}^{\infty} R^{-t}\Gamma(\vec{\theta}, t)] \leq k^0. \tag{6}$$

3.2 Pareto Efficiency Subject to Incentive Compatibility

The planner's problem is to look for an allocation $\vec{\Gamma}$ such that

$$max_{\vec{\Gamma} \in S} U(\vec{\Gamma}, \vec{\theta}, 0) \tag{7}$$

where $\vec{\Gamma}$ satisfies the feasibility condition (6).

Since the cardinality of the set of agents' reporting histories grows exponentially with the number of periods t, the maximization problem (7) is solved in its recursive form. Suppose that each agent is virtually endowed some capital goods. Call it as the claim of capital. The claim of capital is publicly observable. In each period, an agent first makes a report about his/her preference type in the current period and then he/she receives a message which includes consumption today as well as the claim of capital tomorrow. Thus, the claim of capital[9] records the reporting history of an agent.
Formally, let $V^*(k)$ be the value function of an agent with k as the initial claim of capital. For any $k > 0$,

$$V^*(k) = \sup_{\vec{\Gamma} \in S} E_{\vec{\theta}}[\sum_{t=0}^{\infty} \delta^t(\theta_t u(\Gamma(\vec{\theta}, t)))], \tag{8}$$

where $\vec{\Gamma} \in S$ satisfies the temporary feasibility condition

$$E_\theta[\sum_{t=1}^{\infty} R^{-t}\Gamma(\vec{\theta}, t)] \leq k. \tag{9}$$

Given the initial claim of capital k and preference type θ^i, C^i_k is the current consumption level and K^i_k is the claim of capital in the next period[10]. Let $V(\cdot)$ be the expected discounted utility function for future, then the temporary incentive compatibility constraint

[9] There does not exist a market such that agents in the economy may trade on their claims of capital. This is because agents with different preference types will not agree on the price of the claims of capital. See Atkeson and Lucas (1992) for details of the possibility of decentralization and the rational works in this model as well.

[10] Note that agents may make counterfactual preference report, however, if the instantaneous arrangement is temporarily incentive compatible, agents would choose to report truthfully.

(4) can be rewritten with the claim of capital k

$$\theta^i u(C_k^i) + \delta V(K_k^i) \geq \theta^i u(C_k^j) + \delta V(K_k^j), \tag{10}$$

where $i \neq j \in \{1, 2, \ldots, N\}$.
Let T be a functional operator which is defined on the space of real valued functions[11], $\mathcal{F} = \{V(\cdot)|V : \mathcal{R}_{++} \to \mathcal{R}\}$

$$T(V)(k) = \sup_{\{C_k^i, K_k^i\}_{i=1}^N} \sum_i p^i(\theta^i u(C_k^i) + \delta V(K_k^i)), \tag{11}$$

subject to the feasibility condition,

$$\sum_i p^i(C_k^i + K_k^i) \leq Rk, \tag{12}$$

and the temporary incentive compatibility condition,

$$\theta^i u(C_k^i) + \delta V(K_k^i) \geq \theta^i u(C_k^j) + \delta V(K_k^j), \tag{13}$$

for any $k > 0$, $\theta^i \neq \theta^j \in M$.

Lemma 3.2 *If the value function V^* as defined in equation (8) exists, then it satisfies the functional equation $T(V^*) = V^*$. Moreover, V^* is the maximal fixed point of the functional mapping T.*

Find the details of the proof in Appendix.

Define $\widetilde{V}$ to be the value function yielded by the allocation $\vec{\Gamma}$ described as follows: given the current capital level k, it assigns an agent $(R-1)k$ as current consumption and k as the claim of capital in the next period regardless of agents' preference type. Thus $\vec{\Gamma}$ must be incentive compatible and feasible according to the idealized law of large numbers. In addition, $\widetilde{V}$ can be computed analytically: $\widetilde{V}(k) = \sum_{t=0}^{\infty} \delta^t u((R-1)k)$. Since $\vec{\Gamma}$ defined above is feasible and incentive compatible, thus $\widetilde{V}(k) \leq T(\widetilde{V})(k) \leq V^*(k)$.

On the other hand, define $\hat{V}^*$ to be the value function induced by the full risk sharing allocation. It is a natural upper bound of the value function $V^*(k)$.

Let $\mathcal{F}^b$ be a subset of $\mathcal{F}$, $\mathcal{F}^b = \{V \in \mathcal{F}|\widetilde{V} \leq V \leq \hat{V}^*\}$. It can be shown that the functional mapping T defined in equation (11) maps the space $\mathcal{F}^b$ to itself, i.e. for any $k > 0$, and $n \in N$, $\widetilde{V}(k) \leq T^n(V) \leq \hat{V}^*(k)$. Moreover, the monotonicity of T

[11]To be noted that, functions $V \in \mathcal{F}$ are not bounded. The contraction mapping theorem does not apply here. Therefore a weak topology is studied here.

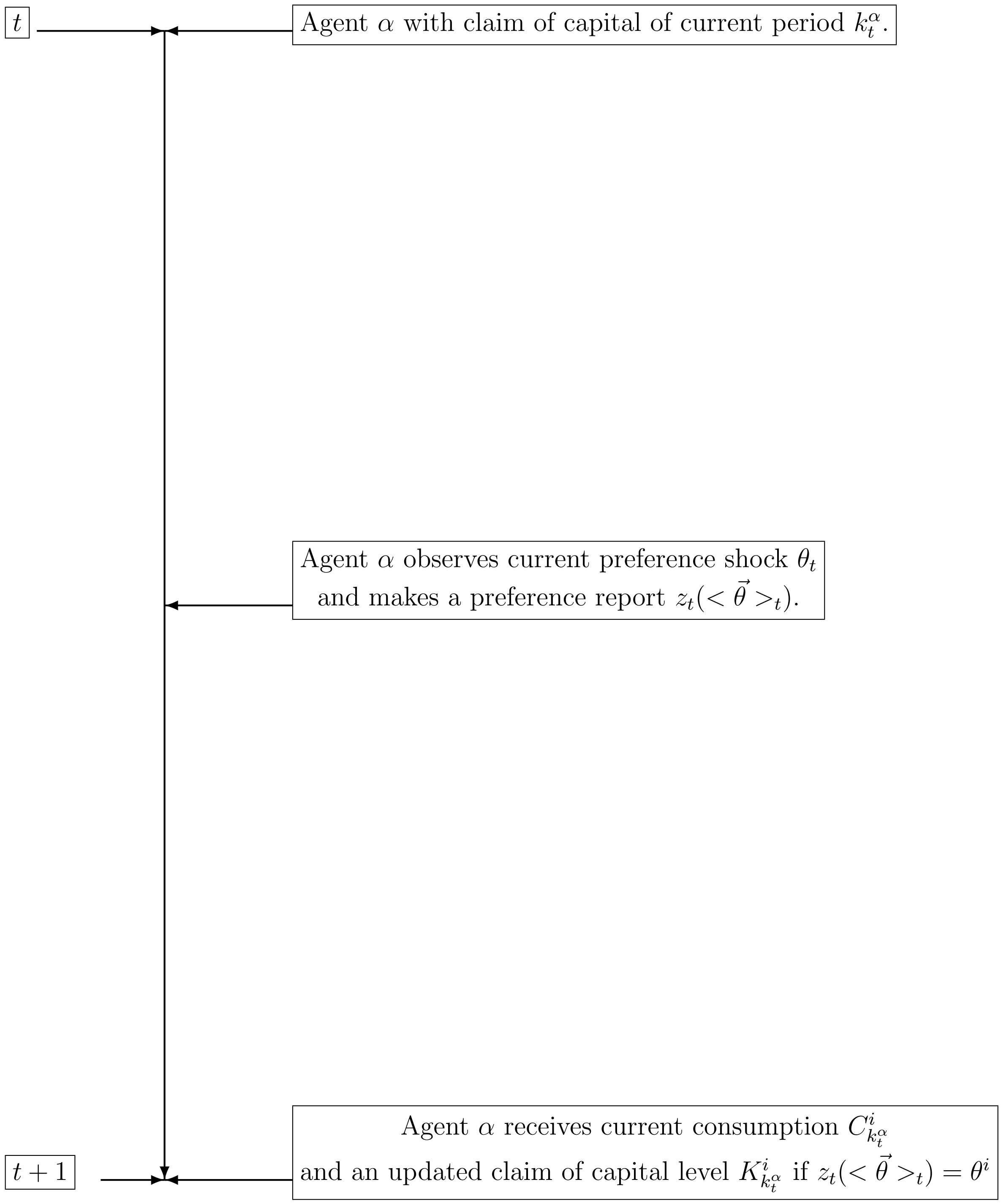

Figure 1: Sequence of events in period t.

implies that $\{T^n(\hat{V}^*)(k)\}_n$ is a monotone decreasing sequence with $\widetilde{V}(k)$ as a lower bound, thus according to the monotone convergence theorem, $\lim_{n\to\infty} T^n(\hat{V}^*)$ exists. Define $V^* \in \mathcal{F}^b$[12],

$$V^* = \lim_{n\to\infty} T^n(\hat{V}^*).$$

In summary:

Lemma 3.3 *For any $k > 0$, if $\lim_{n\to\infty} T^n(\hat{V}^*)(k)$ exists, then define $V^*(k) : (0, \infty) \to \mathcal{R}$ as follows:*

$$V^{**}(k) = \lim_{n\to\infty} T^n(\hat{V}^*)(k). \tag{14}$$

V^* is the maximal fixed point of T on the function space, therefore it equals to the value function defined in equation (7).

See the detail of proof in the Appendix.

According to lemma (3.2), V^* as defined in equation (8) is equal to V^{**}. Therefore, the efficient, incentive compatible allocation can be characterized by solving the problem $T(\hat{V}^{**})$.

Note that the temporary incentive compatibility constraints (13) imply that the current consumption level c_t is positively correlated with the current preference type θ_t.

Lemma 3.4 *For any $k > 0$, if (C_k^i, K_k^i) is temporarily incentive compatible, then $C_k^i > C_k^{i-1}$, and $V(K_k^{i-1}) > V(K_k^i)$, for all $i \in \{1, \ldots, N\}$.*

See the proof in the appendix.

The following lemma states that the global temporary incentive compatibility constraints can be replaced by the local temporary incentive compatibility constraints.

Lemma 3.5 *If both local upward and downward temporary incentive compatibility are satisfied,*

$$\theta^n u(C_k^n) + \delta V(K_k^n) \geq \theta^n u(C_k^{n+1}) + \delta V(K_k^{n+1}),$$
$$\theta^n u(C_k^n) + \delta V(K_k^n) \geq \theta^n u(C_k^{n-1}) + \delta V(K_k^{n-1}),$$

then the global temporary incentive compatibility constraints as defined in equation (3) are satisfied. That is, all nonlocal temporary incentive compatibility constraints are slack.

[12] V^* is used here with a bit abuse of the notation, while in the next lemma I show that V^* is the value function defined by equation (7). Moreover, since $\mathcal{F}^b$ is not a bounded function space, the contraction mapping theorem does not apply here.

Furthermore, if $V \in \mathcal{F}$ is differentiable and strictly concave, then it can be verified that the local upward temporary incentive compatibility constraints are always binding in the optimal case. In turn, the conclusion that any nonlocal upward incentive compatibility constraints and local downward incentive compatibility constraints are slack follows. As a consequence, the $N(N-1)$ temporary incentive compatibility constraints can be replaced by $N-1$ equality constraints.

Lemma 3.6 *If $V \in \mathcal{F}^b$ is differentiable and strictly concave, then the solution to problem (11) satisfies the following statements:*
(1) $(N-1)$ local downward temporary incentive compatibility constraints never bind. That is, for all $n \in \{1, \ldots, N-1\}$,

$$\theta^n u(C_k^n) + \delta V(K_k^n) > \theta^n u(C_k^{n-1}) + \delta V(K_k^{n-1}); \tag{15}$$

(2) $(N-1)$ local upward temporary incentive compatibility constraints are always binding. That is, for all $n \in \{1, \ldots, N-1\}$,

$$\theta^n u(C_k^n) + \delta V(K_k^n) = \theta^n u(C_k^{n+1}) + \delta V(K_k^{n+1}). \tag{16}$$

See the detail of the proof in appendix.

The following 2 lemmas show that $\hat{V}^*$, the value function of the full risk sharing allocation is increasing and strictly concave in capital level k. And it is differentiable at any $k > 0$. What is more, these properties can be preserved under the functional mapping T.

Lemma 3.7 *The value function $\hat{V}^*$ yielded by the full risk sharing allocation is a differentiable, increasing and strictly concave function.*

Lemma 3.8 *The functional mapping T maps a differentiable, increasing and strictly concave function to a differentiable, increasing and strictly concave function.*

For any $n \in N$, $T^n(\hat{V}^*)$ is a differentiable, increasing and strictly concave function. Thus its pointwise limit function V^* is increasing and concave. Therefore, the strict concavity of the period utility function $u(\cdot)$ implies that there exists a unique solution $(C_k^{i*}, K_k^{i*})_i$ to problem $T(V^*)(k)$. It attains the maximum value of the right hand side of equation (11). In addition, an incentive compatible, efficient allocation can be generated recursively as follows: in any period $t \geq 1$, an agent with the claim of capital k_t makes a report about his/her preference type t^i. The current consumption $C_{k_t}^i$ is assigned along with the claim of capital $k_{t+1} = K_{k_t}^i$.

Theorem 3.9 *$V^*(k)$ is increasing, concave, and differentiable on $(0, \infty)$. Moreover, for any $k > 0$, problem $T(V^*)(k)$ has a unique solution $(C_k^{i*}, K_k^{i*})_i$ which attains the maximum value $V^*(k)$.*

In a pure exchange economy, agents in autarky have no way to insure themselves against preference shocks. However, in an economy with capital, agents are able to use capital to smooth their consumption process according to their realized shocks. To demonstrate pareto improvement from the autarkic case, I analyze consider that each agent is endowed with the capital goods instead of the claim of capital. Suppose that agents have the access to the production technology as well. The following proposition states that the informationally constrained efficient allocation strictly improves agents' ex ante expected discounted utility from autarky, although they are able to adjust their consumption level according to their own preference types by adjusting their capital levels.

Thus, the incentive compatible, pareto optimal allocation does offer a partial insurance against agents' idiosyncratic preference shocks.

Proposition 3.10 *The incentive compatible, pareto optimal allocation strictly improves the agents' expected discounted utility from the autarkic case.*

The detail of the proof is offered in the appendix.

3.3 An Example

Theorem (3.10) can be applied to a broad class of period utility function, this subsection uses a simple example to illustrate the procedure of characterizing an informationally constrained efficient allocation.

Consider a 2-state preference type space $M = \{\theta^l, \theta^h\}$, and a CRRA period utility function $u(\cdot)$, $u(c) = \frac{c^\gamma}{\gamma}$, for $\gamma < 0$.

Since the production technology is constant return to scale and the instantaneous preference is iso-elastic, the capital level k is multiplicate to the value function V^*, $V^*(k) = k^\gamma V^*(1)$, where $V^*(1)$ satisfies the normalized planner's problem defined as follows:

$$\begin{aligned} V^*(1) &= \max_{C_1^l, C_1^h, K_1^l, K_1^h} p(\theta^h \frac{(C_1^h)^\gamma}{\gamma} + \delta (K_1^h)^\gamma V^*(1)) \\ &+ (1-p)(\theta^l \frac{(C_1^l)^\gamma}{\gamma} + \delta (K_1^l)^\gamma V^*(1)), \end{aligned} \tag{17}$$

subject to the normalized temporary feasibility constraint

$$p(C_1^h + K_1^h) + (1-p)(C_1^l + K_1^l) = R. \tag{18}$$

According to lemma (3.6), it is enough to consider allocations that satisfy the binding local upward temporary incentive compatibility constraint,

$$\theta^l \frac{(C_1^l)^\gamma}{\gamma} + \delta (K_1^l)^\gamma V^*(1) = \theta^l \frac{(C_1^h)^\gamma}{\gamma} + \delta (K_1^h)^\gamma V^*(1). \tag{19}$$

Equation (19) implies that agents with preference type θ^l are indifferent with telling the truth or not.

Let ζ and λ be the lagrange coefficients of equality constraints (18) and (19) respectively.

The FOCs with respect to $C_1^h, C_1^l, K_1^h, K_1^l$ are given as follows:

$$
\begin{aligned}
(p\theta^h + \lambda\theta^l)(C_1^h)^{\gamma-1} &= p\zeta; && (20)\\
((1-p)\theta^l - \lambda\theta^l)(C_1^l)^{\gamma-1} &= (1-p)\zeta; && (21)\\
(p+\lambda)\delta V^*(1)(K_1^h)^{\gamma-1}\gamma &= p\zeta; && (22)\\
((1-p)-\lambda)\delta V^*(1)(K_1^l)^{\gamma-1}\gamma &= \zeta(1-p). && (23)
\end{aligned}
$$

Given the set of parameter values $(R, \theta^h, \theta^l, p, \gamma, \delta)$, the normalized problem (17) can be solved numerically.

Example 3.11 *Let* $R = 1.2$*,* $\theta^l = 0.5$*,* $\theta^h = 1.5$*,* $p = 0.5$*,* $\delta = 0.9$ *and* $\gamma = -1$*. The numerical solution to (17) is given by* $V^*(1) = -43.4675$*,* $C_1^h = 0.202$*,* $C_1^l = 0.120$ *and* $K_1^h = 1.0157$*,* $K_1^l = 1.0622$*.*

For comparison, the expected discounted utility value in the autarkic case is $V^A(1) = -43.7331$*,*[13] *which is lower than that yielded by the optimal, incentive compatible contract.*

4 Revisit the Exchange Economy

The model studied in this paper is the same as the one in Atkeson and Lucas (1992) except for that in the model studied in this paper, agents are allowed to transfer goods from one period to another through a constant return to scale production technology. Returns from the production are divided into 2 parts, consumption for now and capital for future. Thus a higher rate of return implies more contemporaneous consumption as well as a higher future investment level. Therefore, if there exists a R_0 such that the aggregate consumption in the economy with capital is constant over time, then the efficient, incentive compatible allocation is feasible in an exchange economy with the period endowment level equal to the aggregate consumption. Since the allocation is efficient in a larger set of incentive compatible allocations, then it must be efficient within the set of feasible and incentive compatible allocations of the exchange economy environment as well. Therefore, the existence of an efficient incentive compatible allocation can be shown

[13] The procedure can be found in the appendix.

as a corollary of the results in section 3.

M is the space of shock types, m is the distribution of θ^i on M, then (M, m) defines a preference shock. Given the discount factor δ and the initial investment level k_0, the aggregate consumption of current period $\sum_i p^i C^i_{k_0}$ and the capital in the next period, $\sum_i p^i K^i_{k_0}$ are functions of R, the rate of return of the production.

Lemma 4.1 *There exists a $R_0 > 1$ such that $\sum_i K^i_{k_0}(R_0) = k_0$.*

Find the details of the proof in appendix.

Denote an exchange economy in Atkeson and Lucas (1992) with the constant level of endowment y, the discount factor β, and the preference shock (M, m) as (y, β, M, m).

Correspondingly, denote an economy with a gross rate of return of production technology R_0, the initial endowment k_0, the discount factor δ, and the preference shock (M, m) as (R_0, k_0, δ, M, m).

Proposition 4.2 *Given (M, m), for any $y_0 > 0$, $0 < \beta < 1$, there exists $R_0, k_0 > 0$, such that the efficient, incentive compatible allocation of the economy with capital (R_0, k_0, δ, M, m) is feasible, and optimal within the set of incentive compatible allocations of the exchange economy (y_0, β, M, m), where $k^0 = \frac{y^0}{R_0 - 1}$, $\delta = \frac{\beta}{1-\beta}$.*

There is no formal proof of this proposition but the logic can be stated as follows. If $\delta = \frac{\beta}{1-\beta}$, then agents in these 2 economies have the same time preference across periods. Given the same preference shock (M, μ), lemma (4.1) shows that there exists a R_0 such that the efficient incentive compatible allocation features a constant aggregate consumption in each period. Moreover, since the production technology is constant return to scale, let $k_0 = \frac{y_0}{R_0 - 1}$, then the aggregate consumption in each period is equal to y_0. Thus this allocation is feasible in the exchange economy (y_0, β, M, m). Since the efficient, incentive compatible allocation is optimal in a larger set of incentive compatible allocations. It should be efficient within the feasible, incentive compatible allocations of the exchange economy. Therefore, the conclusion that there exists an informationally constrained efficient allocation in a pure exchange economy described in Atkeson and Lucas (1992) directly follows from the existence of such an allocation in the economy with capital and lemma (4.1).

5 Comparison of Preference Shocks and Income Shocks

Khan and Ravikumar (2001) describe a similar economy as the one discussed in this paper except for the type of shocks. Instead of preference shocks, they assume that agents are subject to a privately observable income shocks. They demonstrate a way to characterize the equilibrium allocation of the economy and implement it numerically. However, the functional mapping studied is defined on an unbounded function space. Thus contraction mapping theorem does not apply in this model either. Although previous literature such as Green (1987) and Atkeson and Lucas (1992) have theoretically shown the existence of such an allocation in a similar environment, further justification has to be made. As a matter of fact, Green (1987) studies a special period utility function and a special endowment process. These assumptions induce a special bellman equation. It features that a contraction mapping defined on a bounded set. In consequence, the logic of his proof does not work for the model in Khan and Ravikumar (2001). Moreover, since there is no production in Atkeson and Lucas (1992) and agents have no way to carry goods over periods, thus the resource constraint studied in their model is different from the one discussed in Khan and Ravikumar (2001). Therefore, the proof of Atkeson and Lucas (1992) does not directly apply to Khan and Ravikumar (2001).

Nevertheless, the logic of the proof in this paper analogically works in an economy with privately observed income shocks. First, consider a real valued function space bounded by the value function yielded by the autarkic allocation and the one yielded by the full risk sharing allocation. Note that if there exists an incentive compatible efficient allocation, then the value function that it generates is bounded by the above 2 functions. Since both of them might not be bounded, the contraction mapping theorem does not apply here. However, the monotonicity of the functional mapping implies that the function sequence induced by iterating the functional mapping on the upper bound function converges to the maximal fixed point of the functional mapping. As the upper bound function is monotone increasing and strictly convex, and the functional mapping defined by the informationally constrained planner's problem maps a strictly concave, monotone increasing function to a strictly concave, monotone increasing one, thus the maximal fixed point function is a pointwise limit of a sequence of increasing and strictly concave functions. Therefore, it is increasing and concave. Furthermore, due to the strict concavity of period utility function, there exists a unique equilibrium allocation.

5.1 The Existence of An Efficient, Incentive Compatible Allocation

Formally, assume the same economic environment as the one discussed in section 3 except that agents face a privately observed income shock in each period, $z^i \in \{z^1, \ldots, z^N\}$[14]. Denote $V_I^I(k)$ to be the function of an agent's expected discounted utility value with incentive compatibility constraints. Correspondingly, denote the value function of an agent without incentive compatibility constraints as $V_I(k)$, and the value function of an agent

[14] In **?**, they consider a 2-state productivity shock. Here we consider a generalization of their model.

in the autarkic case as V_I^A. As in the preference shock case, the optimal allocation in the autarkic case is incentive compatible and feasible for problem $T(V_I^A)(k)$, therefore, $V_I^A(k) \leq T(V_I^A)(k)$ for any capital level k. Moreover, since the full risk sharing allocation assigns all agents the same level of consumption in each period and the same claim of capital for future, it violates agents' incentive compatibility. Therefore, $T(V_I)(k) < V_I(k)$.

Let $\mathcal{F}$ be the space of real valued functions, $\mathcal{F} = \{V | V : \mathcal{R}_{++} \to \mathcal{R}, \quad V_I^A(k) \leq V(k) < V_I^I(k)\}$. Define a functional mapping $T_I : \mathcal{F} \to \mathcal{F}$ as follows:

$$T_I(V)(k) = \max_{(B_k^i, Y_k^i)_i} \sum_i p^i(u(z^i k + B_k^i) + \delta V_I^I(Y_k^i)) \tag{24}$$

subject to

$$\sum_i p^i(B_k^i + Y_k^i) \;=\; 0, \tag{25}$$
$$u(z^i k + B_k^i) + \delta V(Y_k^i) \;\geq\; u(z^i k + B_k^j) + \delta V(Y_k^j), \tag{26}$$

for all $i \neq j$.

Note that $T_I(V_I)(k) \leq V_I(k)$ and $V_I^A(k) \leq T_I(V_I^A)(k)$. Moreover, given any $f, g \in \mathcal{F}$, $f \leq g$,[15] $T_I(f) \leq T_I(g)$.

Therefore, iterating applying the functional mapping T_I on $V_I(k)$, $V_I^A \leq T_I^n(V_I) \leq T_I^{n-1}(V_I) < \ldots < V_I$, for all $n \in N$. Let

$$V_I^I = \lim_{n\to\infty} T_I^n(V_I). \tag{27}$$

By the same logic as lemma (3.8), V_I^I satisfies the bellman equation as follows:

$$V_I^I(k) = \max_{((B_k^h, Y_k^h),(B_k^l, Y_k^l))} p(u(z^l k + B_k^l) + \delta V_I^I(Y_k^l)) + (1-p)(u(z^h k + B_k^h) + \delta V_I^I(Y_k^h)) \tag{28}$$

subject to the resource constraint (25) and the temporary incentive compatibility constraints (26).

According to theorem 4.8 in Stokey et al. (1989), V_I is a strictly concave, monotone increasing function with a unique solution.

Lemma 5.1 *T_I maps a strictly concave function to a strictly concave one.*

[15]Here $f \leq g$ denotes that $f(k) \leq g(k)$ for any $k > 0$.

This lemma can be shown with the same logic as lemma (3.8). Find the details of the proof in the appendix.

Since V_I is a strictly concave function, according to lemma (5.1), $V_I^I = \lim_n T_I^n(V_I)$ is a concave function. Moreover, the strict concavity of the period utility function implies that there exists a unique allocation that maximizes agents utility subject to incentive compatibility.

5.2 Equivalence between Income Shocks and Preference Shocks

Khan and Ravikumar (2001) consider a similar economy except that agents face a privately observed income shock. And they study the competitive equilibrium. Nevertheless, the informationally constrained efficient allocation in the economy with capital can be equivalently viewed as the outcome of a competitive equilibrium. In fact, suppose that each agent is endowed with k_0 units of goods and they transfer them to a financial intermediary (a bank) at the beginning of the initial period. In later periods, withdrawal can be made to satisfy their consumption needs. Assume that financial intermediaries compete on a competitive market, thus the bank maximizes its depositors' expected discounted utility subject to the resource constraint. Therefore, the problem of the competitive financial intermediary is identical to the planner's problem. Furthermore, given any income shock, there exists a preference shock such that capital grows at the same rate in these 2 economies, and vice versa.

Consider the same period utility function as in Khan and Ravikumar (2001), i.e. CRRA period utility function, $u(c) = \frac{c^\gamma}{\gamma}$, $\gamma < 1$, $\gamma \neq 0$[16].

First study the income shock, $z \in \{z^1, z^2\}$[17], where $z^1 < z^2$. As in the preference shock case, the shock history of an agent is fully reflected by the level of his/her claim of capital, it can be used as the state variable of the value function. On the other hand, agents have an iso-elastic preference and the production technology is constant return to scale, thus the capital level does not affect agents' consumption allocatively. In consequence, it is enough to study $\alpha_I^I = V_I^I(1)$. Denote B_1^i to be current transfer of an agent with $z = z^i$, and Y_1^i to be the corresponding claim of capital in the next period, given capital $k = 1$ at the beginning of period.

[16] The same logic also works for logarithm period utility function, however, we need separate analysis.

[17] The analysis below can be extended to an income shock with finite number of states. For simplicity of illustration, assume that the shock has 2 states.

Let $\Phi : \mathcal{R} \to \mathcal{R}$,

$$\Phi(\alpha_I^I) = \max_{((B_1^1,Y_1^1),(B_1^2,Y_1^2))} p(\frac{(z^1+B_1^1)^\gamma}{\gamma} + \delta\alpha_I^I(Y_1^1)^\gamma) + (1-p)(\frac{(z^2+B_1^2)^\gamma}{\gamma} + \delta\alpha_I^I(Y_1^2)^\gamma), \quad (29)$$

subject to

$$\begin{aligned} p(B_1^1+Y_1^1)+(1-p)(B_1^2+Y_1^2) &= 0; & (30)\\ \frac{(z^1+B_1^1)^\gamma}{\gamma} + \delta\alpha_I^I(Y_1^1)^\gamma &\geq \frac{(z^1+B_1^2)^\gamma}{\gamma} + \delta\alpha_I^I(Y_1^2)^\gamma; & (31)\\ \frac{(z^2+B_1^2)^\gamma}{\gamma} + \delta\alpha_I^I(Y_1^2)^\gamma &\geq \frac{(z^2+B_1^1)^\gamma}{\gamma} + \delta\alpha_I^I(Y_1^1). & (32) \end{aligned}$$

Thus the first order conditions of problem (29) can be derived as follows:

$$\begin{aligned} p(z^1+B_1^1)^{\gamma-1} &= p\zeta^I + \phi^I(z^1+B_1^1)^{\gamma-1} - \lambda^I(z^2+B_1^1)^{\gamma-1}\\ (1-p)(z^2+B_1^2)^{\gamma-1} &= (1-p)\zeta^I - \phi^I(z^1+B_1^2)^{\gamma-1} + \lambda^I(z^2+B_1^2)^{\gamma-1}\\ p\delta(Y_1^1)^{\gamma-1}\gamma\alpha_I^I &= p\zeta^I + \phi^I\delta(Y_1^1)^{\gamma-1}\gamma\alpha_I^I - \lambda^I\delta(Y_1^1)^{\gamma-1}\gamma\alpha_I^I\\ (1-p)\delta(Y_1^2)^{\gamma-1}\gamma\alpha^I &= (1-p)\zeta^I - \phi^I\delta(Y_1^2)^{\gamma-1}\gamma\alpha_I^I + \lambda^I\delta(Y_1^2)^{\gamma-1}\gamma\alpha_I^I, \end{aligned} \quad (33)$$

where $\lambda^I, \phi^I, \psi^I \geq 0$ are the lagrange coefficients of constraints (30), (31) and (32) respectively.

The incentive compatibility constraints[18] imply that $Y_1^2 > Y_1^1$. In addition, since λ^I, ϕ^I and $\psi^I \geq 0$, thus $\psi^I = 0$ and $\phi^I > 0$.

Analogously, consider a 2-state preference shock $\theta \in \{\theta^h, \theta^l\}$, where $\theta^l < \theta^h$. For the same reason as in the previous case, it is enough to consider $\alpha^I = V^I(1)$, C_1^i, the current consumption level of an agent with $\theta = \theta^i$, and K_1^i, his/her claim of capital in the next period correspondingly.

Recall the FOCs of problem (17), given by equation (20), (21), (22) and (23),

$$\begin{aligned} p\theta^h(C_1^h)^{\gamma-1} &= p\lambda + \phi\theta^h(C_1^h)^{\gamma-1} - \psi\theta^h(C_2^l)^{\gamma-1}\\ 1-p)\theta^l(C_1^l)^{\gamma-1} &= (1-p)\lambda - \phi\theta^l(C_1^l)^{\gamma-1} + \psi\theta^l(C_1^l)^{\gamma-1}\\ p\delta(K_1^h)^{\gamma-1}\gamma\alpha^I &= p\lambda + \phi\delta(K_1^h)^{\gamma-1}\gamma\alpha^I - \psi\delta(K_1^h)^{\gamma-1}\gamma\alpha^I\\ (1-p)\delta(K_1^l)^{\gamma-1}\gamma\alpha^I &= (1-p)\lambda - \phi\delta(K_1^l)^{\gamma-1}\gamma\alpha^I + \psi\delta(K_1^l)^{\gamma-1}\gamma\alpha^I. \end{aligned} \quad (34)$$

[18] See Khan and Ravikumar (2001) for explanation.

Let $K_1^h = Y_1^1$, and $K_1^l = Y_1^2$. And the above FOC system (33) and (34) are equivalent if,

$$\begin{aligned}
\theta^h (C_1^h)^{\gamma-1} &= (z^1 + B_1^1)^{\gamma-1} c; && (35)\\
\theta^l (C_1^l)^{\gamma-1} &= (z^2 + B_1^2)^{\gamma-1} c; && (36)\\
\theta^l (C_1^h)^{\gamma-1} &= (z^2 + B_1^1)^{\gamma-1} c; && (37)\\
c &= \frac{p\theta^h (C_1^h)^{\gamma} + (1-p)\theta^l (C_1^l)^{\gamma}}{p(z^2 + B_1^2)^{\gamma} + (1-p)(z^1 + B_1^1)^{\gamma}}; && (38)\\
\lambda &= \lambda_I; && (39)\\
\frac{\zeta}{\zeta_I} &= \frac{\alpha}{\alpha_I}, && (40)\\
p\theta^h + (1-p)\theta^l &= pz^h + (1-p)z^l = 1. && (41)
\end{aligned}$$

Given the parameter values z^1, z^2, and $p = Pr(z_t = z^1)$, the solution to problem (29) $((B_1^1, Y_1^1), (B_1^2, Y_1^2))$ is uniquely determined. In turn, the value of $\theta^h, \theta^l, C_1^h$ and C_1^l can be derived according to equations above. Since given θ^h, θ^l, p, the problem (17) has a unique solution which satisfies the FOCs (34) . Therefore, C_1^h, C_1^l, K_1^h, K_1^l compose the unique solution to problem (17) with $Pr(\theta = \theta^h) = p$, and $Pr(\theta = \theta^l) = 1 - p$.

On the other hand, given the parameter values θ^l, θ^h and $p = Pr(\theta_t = \theta^h)$, the solution to problem (17) $((C_1^h, K_1^h), (C_1^l, K_1^l))$ is uniquely determined, thus the value of z^1, z^2, B_1^1, and B_1^2. In addition, $((B_1^1, Y_1^1), (B_1^2, Y_1^2))$ compose the unique solution to problem (29).

6 Conclusion

This paper reformulates the Atkeson and Lucas (1992) model by introducing capital. With a more general period utility function, I show the existence of a unique truth revealing, pareto optimal allocation in the informationally constrained economy with capital. The existence of capital modifies the feasibility constraint of the economy and thus the planner's problem as well. The characterization procedure is demonstrated with a simple model economy.

Besides, I show that the existence of the efficient, incentive compatible allocation of the endowment economy can be verified as a corollary of the results in this paper. Even more, I show that the logic of the proofs can be used in an informationally constrained economy with income shocks as well. Finally, there exists a one-to-one mapping from an income shock process to a preference shock process such that capital grows at the same rate in these 2 economies.

Another observation of the incentive compatible, pareto optimal allocation characterized in this paper is that it makes agents strictly better off from the autarkic case. Since the production technology is constant return to scale, this pareto improvement is not due to the economy of scale, but because of the improvement in risk sharing among agents. Therefore it is of interest to implement it as an equilibrium result of some mechanism. With the same message space, a direct mechanism could be a natural candidate. According to the revelation principle, the efficient, incentive compatible allocation can be supported as a Bayesian Nash equilibrium result. However, how to design a direct mechanism such that it can be uniquely implemented is of particular interest for further research.

7 Appendix

7.1 Proofs

Proof of lemma (3.1).

Define a consumption plan $\vec{\Gamma}$ to be $(\vec{\theta}, t)$ incentive compatible if an agent truthfully reports a $t-$period preference history $(\vec{\theta}, t)$, then it is optimal for him/her to tell the truth thereafter. Denote a $(\vec{\theta}, t)$ incentive compatible allocation as $[(\vec{\theta}, t) - i.c.]$. Formally, $\vec{\Gamma}$ is $[(\vec{\theta}, t) - i.c.]$ if

$$U(\vec{\Gamma}, \vec{\theta}, t) = \max\{U(\vec{\Gamma} \circ \vec{z}, \vec{\theta}, t) | < \vec{z}(\vec{\theta}) >_t = < \vec{\theta} >_t\}.$$

In particular, if a consumption plan is incentive compatible,

$$U(\vec{\Gamma}, \vec{\theta}, 0) = \max\{U(\vec{\Gamma} \circ \vec{z}, \vec{\theta}, 0) | \vec{z} \in Z\},$$

then it is $[(\vec{\theta}, 0) - i.c.]$.

Proof According to the principle of optimality as in Bellman(1957), if an allocation $\vec{\Gamma}$ is incentive compatible, then given any node $(\vec{\theta}, t)$, it is $[(\vec{\theta}, t) - i.c.]$. Moreover, it is both $[(\vec{\sigma}^l_{\vec{\theta},t}, t+1) - i.c.]$ and $[(\vec{\sigma}^h_{\vec{\theta},t}, t+1) - i.c.]$. Hence it is temporarily incentive compatible at $(\vec{\theta}, t)$.

For the inverse direction, it can be shown by contradiction.

Suppose there exists a consumption plan $\vec{\Gamma}$, which is temporarily incentive compatible at any node $(\vec{\theta}, t)$, and satisfies the transversality condition. However, it violates the incentive compatibility condition (3). Thus there exists a reporting strategy $\vec{z}$ such that $U(\vec{\Gamma} \circ \vec{z}, \vec{\theta}, 0) > U(\vec{\Gamma}, \vec{\theta}, 0)$.
Consider a reporting strategy $\vec{z}^n$ which is defined as follows:

$$\vec{z}^n(\theta) = \begin{cases} z_t(< \vec{\theta} >_t), & \text{if } t < n; \\ \vec{\theta}_t, & \text{if } t \geq n. \end{cases}$$

Thus

$$U(\vec{\Gamma} \circ \vec{z}^n, \vec{\theta}, 0) = E_\theta \left[\sum_{t<n} \delta^t \theta_t u(\Gamma(\vec{\theta} \circ \vec{z}, t)) \right] + \delta^n E_\theta V(\vec{\Gamma}, \vec{\theta}, n).$$

Since $\vec{\Gamma}$ satisfies the transversality condition (5), let $t \to \infty$,

$$\lim_{n\to\infty} U(\vec{\Gamma} \circ \vec{z}^n, \vec{\theta}, 0) = U(\vec{\Gamma} \circ \vec{z}, \vec{\theta}, 0).$$

Choose $n = \min\{m | U(\vec{\Gamma} \circ \vec{z}^m, \vec{\theta}, 0) > U(\vec{\Gamma}, \vec{\theta}, 0)\}$.

Assume $t = \max\{\tau | z_\tau(<\vec{\theta}>_\tau) \neq \theta_\tau\}$. Consider another reporting strategy $\vec{z'}$, for all $\vec{\theta} \in M^\infty$, $z'_r(<\vec{\theta}>_r) = z^n_r(<\vec{\theta}>_r)$ for any $r \neq t$, $z'_t(<\vec{\theta}>_t) = \theta_t$. Since $\vec{\Gamma}$ is temporarily incentive compatible at any node $(\vec{\theta}, s)$, $U(\vec{\Gamma} \circ \vec{z'}, \vec{\theta}, s) \geq U(\vec{\Gamma} \circ \vec{z}^n, \vec{\theta}, s)$, which is a contradiction. □

Proof of lemma (3.2):
Proof For any given $k > 0$, first to show $V^*(k) \geq T(V^*)(k)$.
Suppose not, then there exists some $k > 0$, and $\varepsilon_0 > 0$ such that $T(V^*)(k) - V^*(k) > \varepsilon_0$.

According to the definition of $T(V^*)$, there exist $(C^i_k, K^i_k)^N_{i=1}$ such that the temporary incentive compatibility constraints are not binding and

$$T(V^*)(k) - \left(\sum_{i=1}^N p^i \theta^i u(C^i_k) + \delta V^*(K^i_k) \right) < \frac{\varepsilon_0}{2}.$$

Consider an allocation $\vec{\Gamma}$ as follows: in the first period, it assigns agents who report θ^i with C^i_k as contemporaneous consumption.
Let

$$\varepsilon = \min\{\frac{\varepsilon_0}{2}, \varepsilon_{i,j}, \text{for all } i \neq j \in \{1, \ldots, N\}\},$$

where $\varepsilon_{i,j} = \theta^i u(C^i_k) + \delta V(K^i_k) - \theta^i u(C^j_k) - \delta V(K^j_k)$. According to the definition $V^*(\cdot)$, for each i, there exists an incentive compatible allocation $\vec{\widetilde{\Gamma}}^i$ which is feasible with initial capital K^i_k and satisfies $U(\vec{\widetilde{\Gamma}}^i, \vec{\theta}, 0) > V^*(K^i_k) - \frac{\varepsilon}{2}$. For all $t \geq 1$, let $\Gamma(\vec{\theta}, t+1) = \widetilde{\Gamma}^i(\vec{\theta}, t)$ if $\theta_1 = \theta^i$. Since $\vec{\widetilde{\Gamma}}^i$ is temporarily incentive compatible at any node $(\vec{\theta}, t)$ for all $t > 1$, so is $\vec{\Gamma}$. When $t = 1$, according to the definition of $\vec{\widetilde{\Gamma}}^i$,

$$\theta^i u(C^i_k) + \delta U(\vec{\widetilde{\Gamma}}^i, \vec{\theta}, 0) - \theta^i u(C^j_k) - \delta U(\vec{\widetilde{\Gamma}}^j, \vec{\theta}, 0) > 0,$$

for all $i, j \in \{1, \ldots, N\}$. Thus it satisfies the temporary incentive compatible constraints at any node $(\theta, 0)$. In addition, since $\vec{\widetilde{\Gamma}}^i$ satisfies the transversality condition (5), $\vec{\Gamma}$ satisfies the transversality condition (5) as well. Therefore according to lemma (3.1), $\vec{\Gamma}$ is an incentive compatible allocation.

However, the expected discounted utility yielded by allocation $\vec{\Gamma}$ is larger than $V^*(k)$,

$$E_{\vec{\theta}}[\sum_{t=1}^{\infty} \delta^t \theta_t u(\Gamma(\vec{\theta}, t))] - V^*(k) > \frac{\varepsilon_0}{4},$$

which is a contradiction to the definition of V^*.

Next to show that for any $k > 0$, $V^*(k) \leq T(V^*)(k)$.
Suppose not, then there exits some $k > 0$, and $\varepsilon_0 > 0$ such that $V^*(k) - T(V^*)(k) > \varepsilon_0$.

According to the definition of V^*, there exists an allocation which satisfies

$$V^*(k) - E_{\vec{\theta}}[\sum_{t=1}^{\infty} \delta^t \theta_t u(\Gamma(\vec{\theta}))] < \frac{\varepsilon_0}{2}.$$

Let $C_k^i = \Gamma(\vec{\sigma}_{\vec{\theta},0}^i, 1)$, and $K_k^i = E_{\vec{\theta}}[\sum_{t=1}^{\infty} R^{-t}\Gamma(\vec{\sigma}_{\vec{\theta},t+1}^i)|\theta_1 = \theta^i]$. The feasibility and incentive compatibility of $\vec{\Gamma}$ imply that $(C_k^i, K_k^i)_{i=1}^N$ are feasible and temporarily incentive compatible in the initial period. Therefore,

$$\left(\sum_{i=1}^{N} p^i \theta^i u(C_k^i) + \delta V^*(K_k^i)\right) - T(V^*)(k) > \frac{\varepsilon_0}{2},$$

which is a contradiction to the definition of $T(V^*)$.

If there exists another function V' which satisfies the functional equation $T(V') = V'$, then $V^* \geq V'$. Suppose this not true, then there exists $k_0 > 0$ such that $T(V')(k_0) = V'(k_0) > V^*(k_0)$, then let $\varepsilon_0 > 0$ be small enough such that $V'(k_0) - \varepsilon_0 > V^*(k_0)$.

According to the definition of problem $T(V')$, there exist $(C_{k_0}^i, K_{k_0}^i)_i$ such that

$$\sum_i p^i(\theta^i u(C_{k_0}^i) + \delta V'(K_{k_0}^i)) > V'(k_0) - \varepsilon_0(1 - \varepsilon_0).$$

Define an allocation $\vec{\Gamma}$ as follows: for all i, let $\Gamma(\sigma_{\theta,0}^i, 1) = C_{k_0}^i$. Moreover, for the same reason as before, for all i, there exist $(C_{k^i}^j, K_{K_{k_0}^i}^j)_j$ such that

$$\sum_j p^j(\theta^j u(C_{K_{k_0}^i}^j) + \delta V'(K_{K_{k_0}^i}^j)) > V'(K_{k_0}^i) - \varepsilon_0(1 - \varepsilon_0),$$

$\Gamma(\sigma^j_{\sigma^i_{\theta,0},1}, 2) = C^j_{K^i_{k_0}}$, and so on. Thus the value function generated by $\vec{\Gamma}$ is no less than $V'(k_0) - \varepsilon_0 > V^*(k_0)$, which is a contradiction. This completes the proof. $\square$

Proof of lemma (3.3):
Proof
First to show that T maps the function space $\mathcal{F}^b$ to $\mathcal{F}^b$ itself.
Let $V(\cdot) \in \mathcal{T}$, for any given $k > 0$, $T(V)(k)$ can not be greater than $\hat{V}^*(k)$. Otherwise, it implies that given the initial capital level k, there exists an allocation which yields agents a higher ex ante expected discounted utility than that of the full risk sharing case. It contradicts the definition of $\hat{V}^*$. Moreover, $T(V)(k) \geq \widetilde{V}(k)$ since assigning $(R-1)k$ to all agents regardless of their preference types is feasible and incentive compatible. Therefore, if $V \in \mathcal{F}^b$, then $T(V) \in \mathcal{F}^b$, i.e. $\widetilde{V} \leq T(V) \leq \hat{V}^*$.[19]

Next, it can be verified that a fixed point of T in $\mathcal{F}^b$ can be found by iterating T on $\hat{V}^*$.

Follow the previous argument, $T(\hat{V}^*) \leq \hat{V}^*$. By the monotonicity of T, $T^n(\hat{V}^*) \leq T^{n-1}(\hat{V}^*) \leq \ldots \leq \hat{V}^*$. On the other hand, $T^n(\hat{V}^*) \geq \widetilde{V}$, for all $n \in N$. According to the monotone convergence theorem, for any $k > 0$, $\lim_{n\to\infty} T^n(\hat{V}^*)$ exists. Define

$$V^{**}(k) = \lim_{n\to\infty} T^n(\hat{V}^*)(k). \tag{42}$$

Therefore, $V^{**} = T(V^{**})$ is a fixed point of T in $\mathcal{F}^b$.

Finally, it can be shown that V^{**} defined as in equation (42) is the maximal fixed point of T in $\mathcal{F}^b$.

Suppose not, then T has an another fixed point V_0 such that for some $k > 0$, $V_0(k) > V^{**}(k)$. By monotonicity of T, for some n large enough, $T^n(V_0)(k) > T^n(\hat{V}^*)(k)$, which is a contradiction. $\square$

Proof of lemma (3.4):

Proof Recall the temporary incentive compatibility condition (13). In particular, consider $i = n$, $j = n-1$, then

$$\theta^n u(C^n_k) + \delta V(K^n_k) \geq \theta^n u(C^{n-1}_k) + \delta V(K^{n-1}_k) \tag{43}$$
$$\theta^{n-1} u(C^{n-1}_k) + \delta V(K^{n-1}_k) \geq \theta^{n-1} u(C^n_k) + \delta V(K^n_k) \tag{44}$$

[19]For any $V_1, V_2 \in \mathcal{F}^b$, denote $V_1 \leq V_2$ if for any $k > 0$, $V_1(k) \leq V_2(k)$.

Subtract equation (44) from equation (43),

$$\begin{array}{rl} & \theta^n \left(u(C_k^n) - u(C_k^{n-1})\right) \\ \geq & \delta(V(K_k^{n-1}) - V(K_k^n) \\ \geq & \theta^{n-1} \left(u(C_k^n) - u(C_k^{n-1})\right) \end{array} \tag{45}$$

$\theta^n > \theta^{n-1}$ implies that $u(C_k^n) - u(C_k^{n-1}) > 0$, and $V(K_k^{n-1}) - V(K_k^n) > 0$. Moreover, since $u(\cdot)$ is a strictly monotone increasing function, $C_k^n > C_k^{n-1}$. $\square$

Proof of lemma (3.5):
Proof In fact, if local downward and local upward temporary incentive compatibility are satisfied,

$$\theta^i u(C_k^i) + \delta K(C_k^i) \geq \theta^i u(C_k^{i-1}) + \delta K(C_k^{i-1}),$$

and

$$\theta^{i-1} u(C_k^{i-1}) + \delta K(C_k^{i-1}) \geq \theta^{i-1} u(C_k^i) + \delta K(C_k^i),$$

then

$$\theta^i \left(u(C_k^i) - u(C_k^{i-1})\right) \geq \delta(V(K_k^{i-1}) - V(K_k^i). \tag{46}$$

In addition,

$$\begin{array}{rcl} \theta^i \left(u(C_k^{i-1}) - u(C_k^{i-2})\right) & > & \theta^{i-1} \left(u(C_k^{i-1}) - u(C_k^{i-2})\right) \\ & \geq & \delta(V(K_k^{i-2}) - V(K_k^{i-1}), \end{array} \tag{47}$$

where $i \in \{2, \ldots, N\}$.

Add inequality (46) to inequality (47),

$$\theta^i \left(u(C_k^i) - u(C_k^{i-2})\right) > \delta(V(K_k^{i-2}) - V(K_k^i).$$

Rewrite the inequality above

$$\theta^i u(C_k^i) + \delta V(K_k^i) > \theta^i u(C_k^{i-2}) + \delta V(K_k^{i-2}).$$

It implies that for any $i, j \in \{1, \ldots, N\}$ and $i - j = 2$, the temporary incentive compatibility constraint (13)is slack. Moreover, the above steps can be repeated for all $j < i - 1$. Therefore nonlocal temporary incentive compatibility constraints are slack for all $i, j \in \{1, \ldots, N\}$ and $i - j > 1$.

A similar argument applies for all nonlocal temporary incentive compatibility constraints for all $j - i > 1$. $\square$

Proof of lemma (3.6).

Proof Since V is concave and differentiable, let $L(C_k^1,\ldots,C_k^n,K_k^1,\ldots,K_k^n,\zeta,\lambda_1^2,\ldots\lambda_1^N,\ldots,\lambda_N^1,\ldots\lambda_N^{N-1})$ be the lagrange function,

$$\begin{array}{rl} & L(C_k^1,\ldots,C_k^n,K_k^1,\ldots,K_k^n,\zeta,\lambda_1^2,\ldots\lambda_1^N,\ldots,\lambda_N^1,\ldots\lambda_N^{N-1}) \\ = & \sum_i p^i(\theta^i u(C_k^i)+\delta V(K_k^i))-\zeta(\sum_i p^i(C_k^i+K_k^i)) \\ - & \sum_{i,j,i\neq j}\lambda_i^j(\theta^i u(C_k^i)+\delta V(K_k^i)-\theta^i u(C_k^j)-\delta V(K_k^j). \end{array} \tag{48}$$

The statement (1) and (2) can be verified together. In particular, it is enough to show that when $j=i+1$, $\lambda_i^j>0$, and when $j=i-1$, $\lambda_i^j=0$.

Moreover, according to inequality (45), $\lambda_i^j\lambda_j^i=0$, for all $|i-j|=1$. Consider the following cases:
1) Suppose $\lambda_i^{i+1}=0$, and $\lambda_{i+1}^i=0$, then

$$\theta^{i+1}(u(C_k^{i+1})-u(C_k^i))>\delta(V(K_k^i)-V(K_k^{i+1}))>\theta^i(u(C_k^{i+1})-u(C_k^i)).$$

Take the first order derivative of the lagrange function

$$L(C_k^1,\ldots,C_k^n,K_k^1,\ldots,K_k^n,\zeta,\lambda_1^2,\ldots\lambda_1^N,\lambda_N^1,\ldots\lambda_N^{N-1})$$

with respect to K_k^i and K_k^{i+1},

$$\begin{array}{rcl} \delta V'(K_k^i) & = & \zeta; \\ \delta V'(K_k^{i+1}) & = & \zeta. \end{array}$$

Since V is strictly monotone increasing in k, $V'(K_k^i)=V'(K_k^{i+1})$ implies that $K_k^i=K_k^{i+1}$, which is a contradiction.

2) Suppose $\lambda_{i+1}^i>0$, and $\lambda_i^{i+1}=0$, for $i\in\{1,\ldots,N-1\}$, then the first order derivative of the lagrange function

$$L(C_k^1,\ldots,C_k^n,K_k^1,\ldots,K_k^n,\zeta,\lambda_1^2,\ldots,\lambda_i^{i-1},\lambda_i^{i+1},\ldots\lambda_N^{N-1})$$

with respect to K_k^i for $i=1,\ldots,N$

$$\begin{array}{rcl} \delta V'(K_k^1)(1-\dfrac{\lambda_2^1}{p^1}) & = & \zeta; \\ \delta V'(K_k^i)(1-\dfrac{\lambda_{i-1}^i}{p^i}+\dfrac{\lambda_i^{i-1}}{p^i}) & = & \zeta; \quad \text{for } i=2,\ldots,N-1 \\ \delta V'(K_k^N)(1+\dfrac{\lambda_{N-1}^N}{p^N}) & = & \zeta. \end{array}$$

Therefore, $V'(K_k^1)>V'(K_k^N)$. Since V is an increasing and concave function, it implies that $K_k^1<K_k^N$, which is a contradiction.

Therefore, $\lambda_i^j > 0$ if and only if $j = i + 1$. $\square$

Proof of lemma (3.7):
Proof It is straight forward to tell that $\hat{V}^*$ is increasing in capital level k.

For concavity, consider any $k^1, k^2 > 0$. Let $\vec{\Gamma}_{k^i}$ be the first best optimal allocation when the initial capital is k^i. Then for any $\alpha \in (0, 1)$ $\vec{\Gamma}_k = \alpha\vec{\Gamma}_{k^1} + (1-\alpha)\vec{\Gamma}_{k^2}$ is a feasible allocation with the initial capital level $k = \alpha k^1 + (1-\alpha)k^2$.

Moreover, since $u(\cdot)$ is a strictly concave function, for all $t > 0$,

$$\alpha u(\Gamma_{k^1}(\vec{\theta}, t)) + (1-\alpha)u(u(\Gamma_{k^2}(\vec{\theta}, t))) \leq u(\alpha\Gamma_{k^1}(\vec{\theta}, t) + (1-\alpha)\Gamma_{k^2}(\vec{\theta}, t)),$$

where equality holds only for $\alpha = 0$ or 1. Thus $\alpha\hat{V}^*(k_1) + (1-\alpha)\hat{V}^*(k_2) \leq \hat{V}^*(\alpha k_1 + (1-\alpha)k_2)$.

For differentiability, choose $\varepsilon > 0$ small enough. For any $k' \in O(k, \varepsilon)$, define

$$W(k') = \sum_i p^i u(\hat{C}_k^{i*} + \frac{k' - k}{N}) + \delta\hat{V}^*(\hat{K}_k^{i*}).$$

Thus $W(k') \leq \hat{V}^*(k')$, for all $k' \in O(k, \varepsilon)$, and the equality holds when $k' = k$.

According to lemma 1 in (Benveniste and Scheinkman (1979)), $\hat{V}^*$ is differentiable at any $k > 0$. $\square$

Proof of lemma (3.8):

Proof It is straight forward to show that $T(\hat{V}^*)(\cdot)$ is increasing in k.

For concavity, consider $k^1, k^2 > 0$ and $\alpha \in [0, 1]$. Let $k = \alpha k^1 + (1-\alpha)k^2$, and $(C_{k^j}^{i*}, K_{k^j}^{i*})$ be the solution to problem $T(V)(k^j)$, for $j \in \{1, 2\}$.

For all $i \in \{1, \ldots, N\}$, choose $(C_k^i, K_k^i)_i$ such that

$$u(C_k^i) = \alpha u(C_{k^1}^{i*}) + (1-\alpha)u(C_{k^2}^{i*}), \tag{49}$$

and

$$V(K_k^i) = \alpha V(K_{k^1}^{i*}) + (1-\alpha)V(K_{k^2}^{i*}). \tag{50}$$

Since both $u(\cdot)$ and $V(\cdot)$ are both strictly concave, then

$$C_k^i \leq \alpha C_{k^1}^{i*} + (1-\alpha)C_{k^2}^{i*},$$

and

$$K_k^i \leq \alpha K_{k^1}^{i*} + (1-\alpha) K_{k^2}^{i*}.$$

Let $\bar{\varepsilon} = Rk - \sum_i p^i (C_k^i + K_k^i)$, then $\bar{\varepsilon} \geq 0$, and the equal sign holds only when $\alpha = 0$ or 1.

Let $\widetilde{C}_k^i = C_k^i + \bar{\varepsilon}$, and K_k^i be the same as before. According to lemma (3.5), only local upward and downward incentive compatibility constraints needs to be checked. According to assumption (2.2), the period utility function $u(\cdot)$ is concave, the local upward temporary incentive compatibility constraint still holds

$$\begin{aligned}\theta^i(u(\widetilde{C}_k^{i+1}) - u(\widetilde{C}_k^i)) &< \theta^i(u(C_k^{i+1}) - u(C_k^i)) \\ &= \delta(V(K_k^i) - V(K_k^{i+1})).\end{aligned}$$

However, $(\widetilde{C}_k^{i+1}, K_k^i)_i$ may violate the local downward temporary incentive compatibility constraint. But since

$$\theta^{i+1}(u(C_k^{i+1}) - u(C_k^i)) > \delta(V(K_k^i) - V(K_k^{i+1})).$$

By the continuity of period utility function $u(\cdot)$, for each i, there exists an $\varepsilon_i^0 > 0$, such that for all $0 < \varepsilon_i < \varepsilon_i^0$,

$$\theta^{i+1}(u(C_k^{i+1} + \varepsilon_i) - u(C_k^i + \varepsilon_i)) \geq \theta^{i+1}(u(C_k^{i+1} + \varepsilon_i^0) - u(C_k^i + \varepsilon_i^0)) \geq \delta(V(K_k^i) - V(K_k^{i+1})).$$

Let $\varepsilon^* = \min\{\bar{\varepsilon}, \{\varepsilon_0^i\}_i\}$, and choose $\hat{C}_k^i = C_k^i + \varepsilon^*$, K_k^i as defined in equation (50).
Therefore $(\hat{C}_k^i, K_k^i)$ satisfies the feasibility condition and both local upward and downward temporary incentive compatibility conditions. Moreover,

$$T(V)(k) \geq \sum_i p^i(\theta^i u(\hat{C}_k^i) + \delta V(K_k^i)) \geq \alpha V(k^1) + (1-\alpha)V(k^2), \tag{51}$$

where both inequalities hold as equalities only when $\alpha = 0$ or 1.
Thus T maps a strictly concave function to a strictly concave one.

For differentiability, the same argument in the proof of lemma (3.7) applies here. Therefore, for any $k > 0$, $T(V)$ is differentiable at k. □

Proof of the proposition (3.9):

Proof It is straightforward to show that $V^*(k)$ is increasing in k.

For the concaveness, according to lemma (3.7), $\hat{V}^*$ is strictly concave, and lemma (3.8) states that T maps a strictly concave function to a strictly concave one. Therefore, for any $n \in N$, $T^n(\hat{V}^*)$ is strictly concave. Therefore, V^*, the pointwise limit function of a sequence of strictly concave functions is concave.

For the differentiability, for any $k_0 > 0$, let $k' \in O(k_0, \varepsilon)$, where $\varepsilon > 0$ but small enough. Denote $(C^{i*}_{k_0}, K^{i*}_{k_0})$ as the solution to problem $T(V^*)(k_0)$. Choose $K^i_{k'} = K^{i*}_{k_0}$, and $C^i_{k'}$ accordingly such that $(C^i_{k'}, K^i_{k'})_i$ satisfy the temporary incentive compatibility and feasibility condition. Define $W(k')$ as follows:

$$W(k') = \sum_i p^i(\theta_i u(C^i_{k'}) + \delta V^*(K^i_{k'})). \tag{52}$$

Thus $W(k') \leq V^*(k')$, for any $k' \in O(k_0, \varepsilon)$, and the equality only holds when $k = k_0$. According to lemma 1 in Benveniste and Scheinkman (1979) $V^*(\cdot)$ is differentiable at any $k_0 > 0$.

The uniqueness of the solution follows from the concavity of V^*. Suppose by the way of contradiction, there exist 2 different solutions to $V^*(k)$, (C^{i*}_k, K^{i*}_k), $(\widetilde{C}^{i*}_k, \widetilde{K}^{i*}_k)$, for any $\alpha \in (0, 1)$, then follow the same argument as in the proof of lemma (3.8), it is feasible to construct a contract such that it yields a higher $V(k)$ than $V^*(k)$, which is a contradiction. □

Proof of proposition (3.10):

To compare the ex ante expected discounted utility, first I study the agents' problem in autarky. Since all agents are identical ex ante, it is enough to consider the representative agent: the agent maximizes his/her ex ante expected discounted utility value with the initial endowment level k^0. Denote k_t as the capital level in each period t.

According to the principle of optimality, the Bellman equation of the agent in the autarkic case can be written as follows:

$$V^A(k) = \max_{C^{i,A}_k, K^{i,A}_k} \sum_i p^i[\theta^i u(C^{i,A}_k) + \delta V^A(K^{i,A}_k)] \tag{53}$$

subject to the individual feasibility constraint in each period,

$$\begin{aligned} 0 &< C^{i,A}_k, \\ 0 &< K^{i,A}_k \\ C^{i,A}_k + K^{i,A}_k &= Rk, \end{aligned} \tag{54}$$

for all $i \in \{1, \ldots, N\}$.

Follow the similar argument as the proof of lemma (3.7), it can be shown that V^A is an increasing, strictly concave and differentiable function.

Let ζ^i be the lagrange coefficient of an agent with current preference type θ^i. Define the lagrange function as follows

$$\begin{array}{rl} & L(C_k^{1,A},\dots,C_k^{N,A},K_k^{1,A},\dots,K_k^{N,A}) \\ = & \sum_i p^i[\theta^i u(C_k^{i,A}) + \delta V^A(K_k^{i,A})] - \sum_i \zeta^i(C_k^{i,A}+K_k^{i,A}). \end{array} \tag{55}$$

Then the first order conditions of the lagrange function (55) as follows: for each $\theta^i \in M$,

$$\theta^i u'(C_k^{i,A}) = \zeta^i; \tag{56}$$

$$\delta V^A(K_k^{i,A}) = \zeta^i. \tag{57}$$

Proof Since the optimal allocation in the autarkic case satisfies individual agent's incentive compatibility and the feasibility condition of the planner's problem, the expected discounted utility induced by the incentive compatible, pareto optimal allocation is at least as high as that in the autarkic case.

Choose $\tilde{C}_k^i = C_k^{i,A} - \frac{\varepsilon}{p^i}$, $\tilde{C}_k^{N-i} = C_k^{N-i,A} + \frac{\varepsilon}{p^{N-i}}$, for all $i \leq \frac{N}{2}$, where $\varepsilon > 0$ is small enough such that $(\tilde{C}_k^i, \tilde{k}_k^i)_i$ satisfies the local temporary incentive compatibility constraints. According to lemma (3.5), it satisfies the global temporary incentive compatibility constraints. Moreover, let $\tilde{K}_k^i = K_k^{i,A}$, for all $i \in \{1,\dots,N\}$. Therefore, $(\tilde{C}_k^i, \tilde{K}_k^i)_i$ satisfy equation (12), the feasibility condition of the planner, but not equation (54), the feasibility condition of individuals.

Moreover,

$$\begin{array}{rcl} \sum_i p^i\theta^i u(\tilde{C}_k^i) & = & \sum_{i=1}^N p^i\theta^i u(C_k^{i,A}) \\ & + & \sum_{i=1}^{\frac{N}{2}} (\theta^i u'(C_k^{i,A}) - \theta^{N-i}u'(C_k^{N-i,A}))\varepsilon + O(\varepsilon^2). \end{array} \tag{58}$$

Recall the first order condition of agents problem in autarkic case (56) and (57),

$$\theta^i u'(C_k^{i,A}) = \delta(V^A)'(K_k^{i,A}). \tag{59}$$

In fact, according to the feasibility condition (54), equation (59) can be rewritten as

$$\theta^i u'(C_k^{i,A}) = \delta(V^A)'(Rk - C_k^{i,A}). \tag{60}$$

Since $u(\cdot)$, and $V^A(\cdot)$ are increasing and strictly concave, $\theta^i > \theta^j$ implies that $C_k^{i,A} > C_k^{j,A}$. Equivalently, $(V^A)'(R - C_k^{i,A}) < (V^A)'(R - C_k^{j,A})$. Thus $\theta^i u'(C_k^{i,A}) > \theta^j u'(C_k^{j,A})$ for all $\theta^i > \theta^j$.

Therefore,

$$\sum_{i=1}^{\frac{N}{2}}(\theta^i u'(C_k^{i,A}) - \theta^{N-i}u'(C_k^{N-i,A})) > 0,$$

which implies that $\tilde{C}_k^i$ yield a higher aggregated utility level of current period than $\tilde{C}_k^i, \tilde{K}_k^i$. Equivalently, according to the idealized law of large numbers, it improves agents expected utility of current period.
In addition, this step can be repeated for all future periods. In conclusion, the planner can strictly improves agents expected discounted utility from the autarkic case. □

Proof of lemma (4.1):
Proof In fact, when $R = 1$, to satisfy current consumption need, the investment level in the next period $\sum_i p^i K_{k_0}^i(1)$ must be less than that of the current period k_0.

On the other hand, when R is large enough, $\sum_i p^i K_{k_0}^i(R) > k_0$. Suppose not, then for all $R > 0$, $(R-1)k_0 < \sum_i p^i C_{k_0}^i(R) < Rk_0$. That is,

$$\lim_{R\to\infty}\sum_i p^i C_k^i(R) = \infty.$$

Moreover, since $C_{k_0}^1(R) \le C_{k_0}^2(R) \le \cdots \le C_{k_0}^N(R)$, then $\lim_{R\to\infty} C_{k_0}^N(R) = \infty$. Claim that for all i, $C_{k_0}^i(R) \to \infty$ as $R \to \infty$.

In fact, recall the FOCs of problem (28),

$$\begin{array}{rcl} p_1 u'(C_{k_0}^1(R))\theta^1 &=& p_1\zeta(R) + \lambda_1^2\theta^1(R)u'(C_{k_0}^1(R)) \\ p_i u'(C_{k_0}^i(R))\theta^i &=& p_i\zeta(R) - \lambda_{i-1}^i(R)\theta^{i-1}u'(C_{k_0}^i(R)) + \lambda_i^{i+1}(R)\theta^i u'(C_{k_0}^i(R)) \quad \text{for } i = 2,\ldots,N-1 \\ p_N u'(C_{k_0}^N(R))\theta^N &=& p_N\zeta(R) - \lambda_{N-1}^N(R)\theta^N u'(C_{k_0}^N(R)) \\ p_1\delta V'(K_{k_0}^1(R)) &=& p_1\zeta(R) + \lambda_1^2(R)\delta V'(K_{k_0}^1(R)) \\ p_i\delta V'(K_{k_0}^i(R)) &=& p_i\zeta(R) - \lambda_{i-1}^i(R)\delta V'(K_{k_0}^i(R)) + \lambda_i^{i+1}\delta V'(K_{k_0}^i(R)) \quad \text{for } i = 2,\ldots,N-1 \\ p_N\delta V'(K_{k_0}^N(R)) &=& p_N\zeta(R) - \lambda_{N-1}^N(R)\delta V'(K_{k_0}^N(R)) \end{array} \tag{61}$$

According to assumption (2.2), if $C_{k_0}^N(R) \to \infty$, $u'(C_{k_0}^N(R)) \to 0$, then according to the FOC system (56) $u'(C_{k_0}^N(R)) = \frac{\zeta(R)}{1+\frac{\lambda_{N-1}^N(R)}{p_N}}$. Therefore, as $R \to \infty$, either $\zeta(R) \to 0$ or $\lambda_{N-1}^N(R) \to \infty$.
If $\zeta(R) \to 0$ as $R \to \infty$, then for each i, $u'(C_k^i(R)) \to 0$ as $R \to \infty$, which implies that $C_k^i(R) \to \infty$.

On the other hand, if $\lambda_{N-1}^N(R) \to \infty$ as $R \to \infty$, then for all $1 < i < N$, $\lambda_{i-1}^i(R) \to \infty$ as $R \to \infty$. Thus, $C_k^i(R) \to \infty$ as $R \to \infty$.

This claim can be verified by the way of contradiction. Define i_0 as follows,

$$i_0 = min\{i | \text{for any } M \in \mathcal{N}, \text{ there exists a } R \text{ large enough, such that } \lambda^i_{i-1}(R) > M.\}.$$

If $2 < i_0 < N$, then there exists some R large enough such that $u'(C^{i_0-1}_k(R)) = \frac{\zeta(R)}{1+\frac{\lambda^{i_0-1}_{i_0-2}(R)}{p^{i_0}} - \frac{\lambda^{i_0}_{i_0-1}(R)}{p^{i_0}}} < 0$, which is a contradiction.

Sum up FOCs with respect to $C^i_k(R)$,

$$\sum_i p^i u'(C^i_k(R)) \geq \zeta(R),$$

where the inequality sign follows from $u'(C^i_k(R)) < u'(C^{i+1}_k(R))$.

On the other hand, since $\sum_i p^i K^i_k(R) < k$, there exists some k_0 such that $K^i_k(R) < k_0$ for all i. Therefore, sum up the FOCs with respect to $K^i_k(R)$,

$$V'(k_0) \leq \sum_i p^i V'(K^i_k(R)) \leq \zeta,$$

where the inequality sign follows from $V'(K^i_k(R)) > V'(K^{i+1}_k(R))$ and the concavity of V. However, as $R \to \infty$,

$$V'(k_0) \leq \sum_i p^i V'(K^i_k(R)) \leq \zeta \leq \sum_i p^i u'(C^i_k(R)) \to 0,$$

which is a contradiction. This completes the proof. $\square$

Proof of lemma (5.1):

Proof We need show that if V is a strictly concave function, then $T(V)$ is a strictly concave function. That is, if for any $k' \neq k'' > 0$, $r \in (0,1)$, $k = rk' + (1-r)k''$, $rV(k') + (1-r)V(k'') \leq V(k)$, then $rT(V)(k') + (1-r)T(V)(k'') \leq T(V)(k)$. And equality holds if and only if $r = 0$ or 1.

In fact, let $(B^i_{k'}, Y^i_{k'})_i$ and $(B^i_{k''}, Y^i_{k''})_i$ be the solution to the maximization problem (24) with investment level k' and k'' respectively. Now consider a solution candidate of that problem with investment level k.

Since $u(\cdot)$ is strictly concave, it is possible to choose $\widetilde{B}^i_k < rB^i_{k'} + (1-r)B^i_{k''}$ such that $u(z^i k + \widetilde{B}^i_k) = ru(z^i k' + \widetilde{B}^i_{k'}) + (1-r)u(z^i k' + \widetilde{B}^i_{k''})$. Similarly, since by assumption V_I is a strictly concave function, there exists a $\widetilde{Y}^i_k < rY^i_{k'} + (1-r)Y^i_{k''}$ such that $V_I(\widetilde{Y}^i_k) =$

$rV_I(\widetilde{Y}^i_{k'}) + (1-r)V_I(\widetilde{Y}^i_{k''})$. Thus $(\widetilde{B}^i_k, \widetilde{Y}^i_k)_i$ satisfies the incentive compatibility constraints (26) since $(B^i_{k'}, Y^i_{k'})_i$ and $(B^i_{k''}, Y^i_{k''})_i$ satisfy their corresponding incentive compatibility constraints. Therefore it is a feasible solution candidate to problem (24) given investment level k. To be noted that

$$\sum_i p^i(\widetilde{B}^i_k + \widetilde{Y}^i_k) < 0,$$

which implies that it is possible to (incentive compatibly) distribute the extra resource to agents[20]. In consequence, $T(V)(k) \geq rT(V)(k') + (1-r)T(V)(k'')$ and equality only holds when $r = 0$ or 1. □

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[20]According to Thomas and Worrall (1990), only the local downward incentive compatible constraints bind. Therefore, there exists a particular way to do that.

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