A Max-min-max Approach for General Moral Hazard Problem*
(Preliminary)

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Abstract

This paper develops a unified solution method for principal-agent problems with moral hazard under a general setting (i.e., multi-task and multi-signal). Our approach utilizes a key feature of the principal-agent model, namely, the conflict of interest between two parties when the output and effort are given. This feature allows us to establish a max-min-max representation of the original problem. Thus, for any implementable action, an optimal contract for it is a stationary point of the Lagrangian that contains three constraints: the individual rationality constraint, the corresponding first-order condition for the incentive compatibility, and one additional inequality constraint that allows the agent to gain utility under targeted action no less than another alternative action (it is called no-jumping constraint). This contract is called the augmented Mirrlees-Holmstrom (AMH) contract, which is used to characterize the optimal contract and solve the problem, regardless of the validity of the first-order approach (FOA). We apply the new characterization to extend the existing criteria for ranking the efficiency of information systems without the FOA. We show that Holmstrom's sufficient statistic criterion remains valid. However, not even Blackwell's condition is sufficient for the ranking of signals. We thus propose several new criteria without the FOA.

Key Words: Maxmin, Principal-agent, Moral hazard, Solution method

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1 Introduction

Moral hazard principal-agent problems oftentimes involve an incentive compatibility (IC) constraint that is infinite-dimensional and a constraint set that is non-convex. The conventional method for solving these problems is to reduce the dimensionality of the IC constraint first. The first-order approach (FOA), for example, replaces the original IC constraint with a relaxed one of an equality constraint, i.e., the first-order condition (Rogerson, 1985; Jewitt, 1988; Sinclair-Desgagne, 1994; Conlon, 2009; Ke, 2011). Another method, developed by Grossman and Hart (1983), reduces the IC constraint to a finite number of linear constraints, by using an elegant transformation when the state of nature is finite.

The FOA is not always valid as it may oversimplify the IC constraint; whereas the Grossman and Hart's method does not work when output is continuous or state of nature is infinite, nor does it provide a general characterization of an optimal contract.

In this paper, we show that, by adding only one more inequality constraint to the first order condition, we can obtain a necessary and sufficient condition for an optimal contract and derive a general characterization of an optimal contract.

To understand how we choose a constraint to be added to the first order condition, let us begin with Mirrlees' (1975 or 1999) original discussion of an example in which there are only two best responses at the optimum. He adds, to the first order condition, an inequality constraint that ensures the agent to gain expected utility under the targeted action no less than another alternative action. This inequality constraint ensures that the agent will not jump to the alternative best response. We therefore refer to it as a no-jumping constraint. Extending Mirrlees' two-best-response example to a more general principal-agent problem in which the number of agent's best responses are assumed to be finite, we can construct a Lagrangian with multiple no-jumping constraints to characterize an optimal contract (see Mirrlees, 1986). In this case, we can reduce the original problem into a Lagrangian dual problem with finite a number of constraints.

Mirrlees' Lagrangian approach involves many first order conditions, including not only the one for the targeted action, but also those for all alternative best responses. The approach ends with having too many Lagrangian multiplers, exceeding the number of equations for the approach to be of practical use. Araujo and Moreira (2001) improved upon this approach. They show that, when the agent's utility is separable and the task is unidimensional, only the first-order condition for the targeted action remains necessary as the rest of the first order conditions will be automatically satisfied at the optimum. Furthermore, by adding the second-order condition as an inequality constraint to the first order condition for the targeted action, they are able to close the gap between the number of unknowns and that of equations in Mirrlees' approach.

However, because Araujo and Moreira (2001) also apply the no-jumping constraints as Mirrlees (1986) does, their approach share a similar logical dilemma as Mirrlees' approach does. That is, in order to apply these no-jumping constraints, one needs to know what are all the best responses to an optimal contract; and yet, without knowing the optimal contract, one are not able to find out what are these best responses in the first place.[1]

Our method avoids the indeterminacy of picking the set of no-jumping constraints, hence does not require having a priori knowledge of, the set of alternative best responses. Instead, it makes use of two observations. First, fixing a targeted action, a principal-agent problem can be reduced to a maxmin game. In this maxmin game, the principal chooses an optimal contract to maximize his payoff, subject to the no-jumping constraint that the agent weakly prefers the targeted action to other best responses, while these other best responses minimizes the principal's maximal payoff through the no-jumping constraint.[2] Second, fixing a targeted action, a principal-agent problem is also a zero-sum game. This allows me to establish equivalence between the previous maxmin game and the following minmax one. In this minmax game, the principal chooses a contract to maximize his payoff subject to the no-jumping constraint that the agent weakly prefers the targeted action to a generic alternative, with the agent choosing one among all the generic alternatives to minimize the principal's maximized payoff. As the equivalence holds for all targeted actions, we are able to choose an optimal target and thus solve the original principal-agent problem. To sum up, we can find a solution of the original moral hazard problem by a max-min-max procedure, where the first "max" is over an action, the second "min" is over an alterantive action, and the third "max" is over a contract.

Our method has two advantages. First, as we have stated above, it does not rely on a priori information about the alternative best responses to an optimal contract. Second, it uses the least necessary conditions for solving the original problem and obtaining the most parsimonious characterization of an optimal contract. Compared with Grossman and Hart (1983), my method is able to deal with problems involving continuous output and infinite states of nature; it allows the conventional Kuhn-Tucker conditions to be applicable in solving for an optimal contract.

While the existing literature has replied upon necessary conditions to bring about some features of an optimal contract, my method establishes necessary and sufficient conditions and hence is able

[1] That is, one has to use trial and error for all the possibilities of having different numbers of best responses.

[2] The agent's alternative best response does not enter the principal's objective function directly, however it can shrink the principal's choice set through the no-jumping constraint. In particular, if the set is shrunk to be empty, we assume the maximum over an empty set to be minus infinity.

to offer the exact characterization of an optimal contract.

This knowledge enables me to establish a set of necessary and sufficient conditions for ranking signals, which in turn allows me to shed lights on some of the established results in the literature. The existing criteria about signal ranking (Holmstrom 1979, Kim 1995, Jewitt 1997, and Xie 2011) assume the validity of the FOA. When the FOA fails, we show that Holmstrom's (1979) sufficient statistic criterion remains valid, while Kim's (1995) criterion needs additional conditions, and in particular Blackwell's condition becomes no longer sufficient (Kim 1995). We show that Blackwell's condition is sufficient for the ranking of signals when signals satisfy the monotone likelihood ratio property (MLRP).

The rest of the paper is organized as follows. We first present the set-up and related research in Section 2. Section 3 details the main results and discusses several of its implications. Section 4 demonstrates the applications in signal ranking and in characterizing an optimal contract. Section 5 generalizes the main result to consider a more general utility structure. Section 6 concludes. Technical proofs are relegated to the Appendix.

2 The Model and the Existing Approaches

2.1 Principal-agent Model

We start with the classical setting of moral hazard with multiple tasks and multipdimensional signals. (The extension will be discussed in Section 5). The agent privately takes an action $a = (a_1, a_2, ..., a_N) \in \mathbb{A} \equiv \prod_{i=1}^{N} [\underline{a}_i, \bar{a}_i]$, which affects the output $X \in \mathcal{X} \subset \mathbb{R}^K$ through a probability density function (p.d.f.) $f(x, a)$. We assume that $f(x, a)$ is continuous and differentiable in a up to the second order and the support of X does not depend on a.

Assume that the agent has a smooth Bernoulli utility function

$$u(w) - c(a),$$

where w is the agent's monetary payoff, $u(.)$ is a strictly increasing and concave function, and $c(.)$ is the cost of action. The principal chooses wage $w = w(x) \geq \underline{w}$ as a function of output x, which is bounded from below by an exogenous minimum wage $\underline{w}$.[3] Following Sinclair-Desgagne (2004) and Conlon (2009a), let the value of output be given by the function $\pi : \mathcal{X} \to \mathbb{R}$. The principal

[3] We assume that the low bound $\underline{w}$ has a property such that $u(\underline{w}, a) > -\infty$ for all $a \in \mathbb{A}$. The reason for having a lower bound of payment is to rule out some Mirrlees' type counter example for non-existence. We do not address the issue of existence in this paper (see Kadan, Reny, and Swinkels (2011) and Ke (2011b) for further discussion on the existence of a solution to (P1)).

has a utility function $v(.)$ over the net value $(\pi - w)$, where $v(.)$ is smooth, strictly increasing, and weakly concave. Thus, the principal's expected utility is

$$V(w,a) = \int v(\pi(x) - w(x))f(x,a)dx$$

and the agent's expected utility is

$$U(w,a) = \int u(w(x))f(x,a)dx - c(a),$$

where we use the short notation w instead of $w(.)$ for simplicity.

The principal faces a maximization problem (assume the existence of a solution)

$$(P1) \qquad \max_{(w,a)} V(w,a),$$

subject to the incentive compatibility (IC) constraint

$$U(w,a) - U(w,a') \geq 0 \text{ for } \forall a' \in \mathbb{A}; \tag{IC}$$

and the individual rationality (IR) constraint

$$U(w,a) \geq \underline{U}, \tag{IR}$$

where $\underline{U} > -\infty$ is the outside reservation utility.

In this paper, we also use notation

$$a \in a^{BR}(w) \equiv \arg\max_{a'} U(w,a')$$

to represent the IC constraint for a. And formally, an action a is said to be *implementable* if $a \in a^{BR}(w)$ for some w such that $U(w,a)$ is bounded and satisfies the IR constraint.

2.2 The First-Order Approach

Note that the IC constraint in (P1) consists of infinitely many inequalities. The first-order approach (FOA) assumes that one can replace the IC constraint by a relaxed IC constraint (RIC). That is, the first-order condition with respect to a_i $(i = 1, 2, ..., N)$:

$$U_{a_i}(w,a) = 0 \text{ if } a_i \in (\underline{a}_i, \bar{a}_i);\ U_{a_i}(w,a) \leq 0 \text{ if } a_i = \underline{a}_i;\text{ and } U_{a_i}(w,a) \geq 0 \text{ if } a_i = \bar{a}_i, \tag{RIC}$$

where the subscripts denote partial derivatives.

When one replaces the IC constraint by the RIC constraint, the optimization problem (P1) becomes

$$\max_{(w,a)}\{V(w,a) : (IR) \text{ and } (RIC)\}. \tag{RP1}$$

The validity of the FOA means that the maximum values of problems (RP1) and (P1) are the same.[4]

For convenience, we use the following notations:

$$\mathcal{F} \equiv \{(w,a) : (IR) \text{ and } (IC) \text{ are satisfied}\}, \text{ and}$$
$$\mathcal{F}^R \equiv \{(w,a) : (IR) \text{ and } (RIC) \text{ are satisfied}\}.$$

Clearly, (RP1) is a routine optimization problem that only involves a set of equality or inequality constraints. However, as (RIC) is necessary for (IC), we have $\mathcal{F} \subset \mathcal{F}^R$; thus, (RP1) may admit some solutions that are unattainable under the original IC constraint. To make the FOA valid, the existing literature has proposed various sets of sufficient conditions (see, e.g., Rogerson, 1985; Jewitt, 1988; Sinclair-Desgagne, 2004; Conlon, 2009) for the agent's expected utility $U(w,a)$ to be globally concave (under some non-decreasing $w(.)$) in a so that (RIC) is necessary and sufficient for (IC). Recently, Ke (2011a) provides a new set of sufficient conditions based on a *fixed-point* method without requiring $U(w,a)$ to be globally concave in a. However, these conditions for the validity of the FOA impose strong restrictions on either the information structure or utility function or both, so the FOA may be invalid for many practical settings. We therefore wonder if there is any general approach to solve (P1) without the FOA.

2.3 The Generalized Lagrangian Approaches and Main Difficulties

Mirrlees (1975) tries to propose a general Lagrangian approach for (P1) when the FOA is invalid.[5] He notes that the issue of the FOA is due to the fact that there are multiple best responses satisfying the local first-order condition (RIC). Then, he introduces an additional inequality constraint

$$U(w,a) - U(w,\hat{a}) \geq 0, \tag{NJ($\hat{a}$)}$$

which is called *no-jumping* (NJ) constraint at $\hat{a}$, since this constaint prevent the agent's action from jumping to $\hat{a}$. The IC constraint is equivalent to a set of NJ constraints for every $\hat{a} \in \mathbb{A}$. The number of NJ constraint can be reduced if there are $r+1$ distinct best responses at the optimum. In this case, if we can find all alternative best responses $\hat{a}^i$ $(i = 1, 2, ..., r)$ such that $U_a(w, \hat{a}^i) = 0$, then we only need consider r NJ constraints. Mirrlees (1986) then proposes a maximization problem

$$\max_{(w,a)} \{V(w,a) : U_a(w,a) = 0,\ U(w,a) \geq \underline{U},\ \text{and } U(w,a) - U(w,\hat{a}) \geq 0 \text{ for all } \hat{a} \text{ s.t. } U_a(w,\hat{a}) = 0\}$$

to represent the original problem, where (NJ($\hat{a}$)) binds at a set of distinct points $\hat{a} = \hat{a}^1, ..., \hat{a}^r$.

[4] In this paper, we call two maximization problems equivalent if the resulting maximum values are the same.

[5] Grossman-Hart approach does not apply when x is continuous.

Compared with the original problem, Mirrlees' representation reduces the infinite-dimensional IC constraint to a finite number of inequality or equality constraints. The detail of the number of constraint is listed as follows

$$\begin{aligned} rN &: \text{first-order condition for alternative best responses} \\ N &: \text{first-order condition for the optimal action} \\ r &: \text{no-jumping constraints} \\ 1 &: \text{IR constraint} \\ (1+r)(N+1) &: \text{In total.} \end{aligned}$$

However, applying Mirrlees' approach entails several difficulties. First, the manner by which these $\hat{a}$ are determined remains open. We need to know $\hat{a}$ before we can solve the problem. In fact, each $\hat{a}^i$ $(i = 1, 2, ..., r)$ and the number of alternative best reponses r depend on the optimal contract w^{**} that has not been solved yet. Determining the number of best responses is difficult in the absence of a full characterization of the optimal contract w^{**}. A very strong smoothness restriction on the expected utility is also required to apply Mirrlees and Robert's (1980) theorem on the number of distinct maxima (the objective function needs to be infinitely differentiable). Second, in some cases, a continuum set of best responses, which is not easy to characterize, may exist. Third, as Araujo and Moreira (2001) point out, Mirrlees' Lagrangian uses all of the first-order conditions for each distinct best response so that the number of unknown variables is greater than the number of equations.

Araujo and Moreira (2001) improve Mirrlees' Lagrangian (only in the case of single task) by removing the RIC constraints for alternative best responses and introducing an additional second-order constraint.[6] Thus, there is no deficit of equations for solving the unknowns. To understand their main idea, let us consider a simple example where only two interior best responses exist against the optimal contract w^{**}. Then, the IC constraint is binding at the two distinct points a^{**} and $\hat{a}^*$. If we *ex ante* know a^{**} and $\hat{a}^*$, the original problem can be solved by

$$\max_{(w,a)}\{V(w,a) : U_a(w,a) = 0,\, U(w,a) \geq \underline{U},\; U_{aa}(w,a) \leq 0 \text{ and } U(w,a)-U(w,\hat{a}) \geq 0 \text{ for } \hat{a} \in \{a^{**}, \hat{a}^*\}\}.$$

Therefore, they construct a Lagrangian function for the above constrained maximization problem and characterize an optimal contract, which uses less constraint than what Mirrlees (1986) does. The idea can be generalized to the situation where there is a Borel set of best responses so that the RIC, NJ, or the second-order constraint is measure-based.

However, Araujo and Moreira's (2001) Lagrangian depends on topological properties of the best response mapping (e.g., connectedness and countability), which seems complicated. One may

[6] Dealing with the second-order constraints becomes more difficult when the action is multidimensional.

wonder if the Lagrangian can be simplified further. More importantly, Araujo and Moreira's (2001) characterization still requires *a priori* information about the best response. Thus, the issue is not resolved because the form of an optimal contract depends on the property of the best response mapping, and which, in turn, depends on the shape of the optimal contract offered.[7] We resolve this issue by showing that the addition of one more no-jumping constraint to the RIC constraint is sufficient to characterize an optimal contract.

3 A General Approach Based on Dual Representation

In this section, we present the main results and provide an explicit rule in characterizing an optimal contract w^{**} and determining an optimal action a^{**} together with an alternative best response $\hat{a}^*$, using only one no-jumping constraint.

3.1 The Characterization with One NJ Constraint

This subsection investigates the property of maximization problem with (NJ($\hat{a}$)) for a given $\hat{a}$. We will show why it is optimal later. Consider a maximization problem over w given $(a, \hat{a})$ and U:

$$\max_{w}\{V(w,a) : U(w,a) \geq U, \text{ (RIC)}, U(w,a) - U(w,\hat{a}) \geq 0\}, \qquad (\text{P}|a,\hat{a};U)$$

where $U \geq \underline{U}$ is a parameter adjusting the agent's utility.

For simplicity, we neglect the corner solutions in which (RIC) is not binding.[8] Thus, we can construct the Lagrangian

$$\mathcal{L}(w,a,\hat{a};\lambda,\mu,\delta;U) = V(w,a) + \lambda[U(w,a) - U] + \mu \cdot U_a(w,a) + \delta[U(w,a) - U(w,\hat{a})], \qquad (1)$$

where $\lambda \geq 0$, $\mu = (\mu_1, \mu_2, ..., \mu_N) \in \mathbb{R}^N$, and $\delta \geq 0$ are the Lagrangian multipliers with respect to constraints $U(w,a) \geq U$, (RIC), and (NJ($\hat{a}$)), respectively. Throughout this paper, by default we mean $\lambda \geq 0$, $\mu = (\mu_1, \mu_2, ..., \mu_N) \in \mathbb{R}^N$, and $\delta \geq 0$ without further specification.

[7] If Mirrlees and Robert's (1980) conclusion about the number of distinct maxima holds, then finding the "fixed point" may be possible in the sense that the number of distinct best responses is the same as the number of no-jumping constraints used in the characterization of the optimal contract. Under some restrictions, the number of constraints that satisfies the fixed-point property is $N+1$, where N is the dimension of a. Roughly speaking, if a is N-dimensional, then one needs to have $N+1$ no-jumping constraints to characterize the optimal contract, and under that contract, the number of best responses may be no more than $N+1$. However, specifying the sufficient condition for the property of the fixed-point number of constraints to hold may be difficult.

[8] If the optimal action for the i-th task is $a_i = \underline{a}_i$ then the i-th RIC constraint for the relaxed problem should be $U_{a_i}(w,a) \leq 0$. If the optimal action is $a_i = \bar{a}_i$, the i-th RIC constraint for the relaxed problem should be $U_{a_i}(w,a) \geq 0$. We can divide the RIC constraints into three sub-groups to deal with the issue, but it does not change the conclusion.

When the support does not depend on the action as we assume, we can further write the Lagrangian as

$$\mathcal{L}(w, a, \hat{a}; \lambda, \mu, \delta; U) = \int L(w, a, \hat{a}; \lambda, \mu, \delta; U) f(x, a) dx,$$

where

$$\begin{aligned} & L(w, a, \hat{a}; \lambda, \mu, \delta; U) \\ = \quad & v(\pi - w) + \lambda(u(w) - c(a) - U) + \mu \cdot [u(w) l_a(x, a) - c_a(a)] + \delta[u(w)(1 - \frac{f(x, \hat{a})}{f(x, a)}) - c(a) + c(\hat{a})] \end{aligned}$$

with $l_a(x, a) = \frac{\partial \log f(x,a)}{\partial a}$. The maximization of $\mathcal{L}(w, a, \hat{a}; \lambda, \mu, \delta; U)$ over w can be done pointwise through $L(w, a, \hat{a}; \lambda, \mu, \delta; U)$. The concavities of $v(.)$ and $u(.)$ imply that $L(w, a, \hat{a}; \lambda, \mu, \delta; U)$ is single-peaked in w given every other arguments. Therefore, there exists a unique w such that the first-order condition $\frac{\partial}{\partial w} L(w, a, \hat{a}; \lambda, \mu, \delta; U) = 0$ is satisfied for $w \geq \underline{w}$, or $w = \underline{w}$ if $\frac{\partial}{\partial w} L(w, a, \hat{a}; \lambda, \mu, \delta; U) < 0$. This unique w can be solved by the equation (for almost every x)

$$\frac{v'(\pi - w)}{u'(w)} = \lambda + \mu \cdot l_a(x, a) + \delta(1 - \frac{f(x, \hat{a})}{f(x, a)}), \tag{2}$$

whenever $\lambda + \mu \cdot l_a(x, a) + \delta(1 - \frac{f(x,\hat{a})}{f(x,a)}) \geq \frac{v'(\pi - \underline{w})}{u'(\underline{w})}$. We denote the solution by $w_{\lambda,\mu,\delta}(x, a, \hat{a})$. Since it plays a central role in our analysis, we formally define this contract as follows.

Definition 1 *A contract w is called augmented Mirrlees-Holmstrom (AMH) contract if it satisfies the first-order condition (2) for some parameter $(\lambda, \mu, \delta, \hat{a}, a)$.*

The main result of this paper will show that regardless of the validity of the FOA, there exists an optimal contract that is AMH contract.[9] Note that without the item $\delta(1 - \frac{f(x,\hat{a})}{f(x,a)})$, (2) returns to the classic Mirrlees-Holmstrom (MH) condition, which characterizes the MH contract. So without the FOA, an optimal contract only has one extra term, compared with the MH contract. This term takes care of the NJ constraint.

There are five parameters $(\lambda, \mu, \delta, \hat{a}, a)$ in an AMH contract $w_{\lambda,\mu,\delta}(x, a, \hat{a})$. In the rest of this subsection, Lemma 1 shows that there is a unique Lagrangian multiplier vector (λ, μ, δ) solving the three constraints of problem (P$|a, \hat{a}; U$), in terms of $(\hat{a}, a)$ and U. From next subsection on, we will figure out how to determine $(\hat{a}, a)$ and U so that there is an optimal AMH contract solving the original problem (P1).

Lemma 1 *If $u(.)$ is concave and $v(.)$ is weakly concave, then for every $(a, \hat{a})$ and a solution w^* of (P$|a, \hat{a}; U$), there exists a unique multiplier vector $(\lambda^*, \mu^*, \delta^*)$ such that (i) for almost every x, the*

[9] Having $w \to \infty$ with a positive probability measure is impossible. Therefore, $w_{\lambda,\mu,\delta}(x, a, \hat{a})$ is bounded almost everywhere.

first-order condition (2) holds for w^* *whenever* $w^* \geq \underline{w}$*; (ii) the complementary slackness conditions hold:*

$$\begin{array}{rrcl} (ii\text{-}a) & \lambda^* & \geq & 0,\ U(w^*,a)-U \geq 0 \text{ and } \lambda^*[U(w^*,a)-U]=0, \\ (ii\text{-}b) & 0 & = & U_a(w^*,a), \quad \text{and} \\ (ii\text{-}c) & \delta^* & \geq & 0,\ U(w^*,a)-U(w^*,\hat{a}) \geq 0 \text{ and } \delta^*[U(w^*,a)-U(w^*,\hat{a})]=0; \end{array}$$

and (iii) $(\lambda^*,\mu^*,\delta^*) = (\lambda^*(a,\hat{a};U),\mu^*(a,\hat{a};U),\delta^*(a,\hat{a};U))$ *depends on* $(a,\hat{a};U)$ *in a continuous and differentiable almost everywhere (a.e.) manner (see Appendix A1 for the proof).*

Based on Lemma 1, we can write the solution of $(\mathrm{P}|a,\hat{a};U)$ as

$$w^*(.,a,\hat{a};U) = w_{\lambda^*(a,\hat{a};U),\mu^*(a,\hat{a};U),\delta^*(a,\hat{a};U)}(.,a,\hat{a}), \tag{3}$$

where $(\lambda^*(a,\hat{a};U),\mu^*(a,\hat{a};U),\delta^*(a,\hat{a};U))$ is the Lagrangian multiplier vector. The value function given $(a,\hat{a})$ is

$$V(w^*(.,a,\hat{a};U),a) = \min_{(\lambda,\mu,\delta)} \max_{w} \mathcal{L}(w,a,\hat{a};\lambda,\mu,\delta;U) = \mathcal{L}^*(a,\hat{a};U),$$

where we use notation

$$\mathcal{L}^*(a,\hat{a};U) \equiv \mathcal{L}(w^*(.,a,\hat{a};U),a,\hat{a};\lambda^*(a,\hat{a};U),\mu^*(a,\hat{a};U),\delta^*(a,\hat{a};U);U). \tag{4}$$

By the theorem of maximum, $\mathcal{L}^*(a,\hat{a};U)$ is continuous in $(a,\hat{a})$ because both the maximizer of the Lagrangian over w and the Lagrangian multiplier vector are unique.

3.2 Important Inequalities

In this subsection, we investigate the relationship between $(\mathrm{P}|a,\hat{a};U)$ and the original problem. Our goal is to show the equivalence between them when $(a,\hat{a})$ and U are sophisticatedly chosen.

For convenience, denote a constraint set with a particular NJ constraint as

$$\mathcal{S}_{\hat{a}} \equiv \{(w,a) : \text{(IR), (RIC), and (NJ}(\hat{a}))\}. \tag{5}$$

We introduce several problems based on constraint set $\mathcal{S}_{\hat{a}}$ or its analogues.

First, note that $U(w,a)-U(w,\hat{a})$ is minimized when $\hat{a} \in a^{BR}(w)$. Treating the principal-agent problem as a leader-follower game, given the principal's choice (w,a), the agent may deviate to choose $\hat{a}$ so that $(\mathrm{NJ}(\hat{a}))$ becomes more difficult to be satisfied. In particular, if $\mathcal{S}_{\hat{a}}$ is empty, we assume that the maximum over an empty set is minus infinity. The principal as a leader has to choose an incentive compatible $(w,a) \in \mathcal{F}$ to avoid the emptiness of the constraint set. Therefore, the original problem is equivalent to

$$\max_{(w,a)} \inf_{\hat{a}} \{V(w,a) : (w,a) \in \mathcal{S}_{\hat{a}}\}. \tag{P1$'$}$$

Problem (P1′) only uses one no-jumping constraint. However, it does not seem helpful in terms of characterizing an optimal contract, because the correspondence $a^{BR}(w)$ is not continuous and the Lagrangian method may not be applicable. We then introduce a new problem by switching the order of optimization

$$\max_{a} \left(\inf_{\hat{a}} \max_{w} \{V(w,a) : (w,a) \in \mathcal{S}_{\hat{a}}\} \right). \tag{P2}$$

In problem (P2), because w is chosen after $\hat{a}$ is chosen, problem (P2) may not be equivalent to problem (P1′). Indeed, by the minmax inequality,

$$\max_{a} \left(\inf_{\hat{a}} \max_{w} \{V(w,a) : (w,a) \in \mathcal{S}_{\hat{a}}\} \right) \geq \max_{(w,a)} \inf_{\hat{a}} \{V(w,a) : (w,a) \in \mathcal{S}_{\hat{a}}\},$$

therefore, problem (P2) can be regarded as a sharper relaxed problem of (P1) than (RP1).

When problems (P2) and (P1′) are not equivalent, we want to narrow down the choice set even further. Let

$$U^* = U(w^{**}, a^{**})$$

be the agent's utility at the optimum (w^{**}, a^{**}), then the following constraint

$$U(w,a) \geq U^*, \tag{IR*}$$

is binding at the optimum. We replace (IR) by (IR*) and obtain a smaller constrained set

$$\mathcal{S}^*_{\hat{a}} \equiv \{(w,a) : (\text{IR}^*), (\text{RIC}), \text{ and } (\text{NJ}(\hat{a}))\}. \tag{6}$$

As $U^* \geq \underline{U}$, $\mathcal{S}^*_{\hat{a}} \subset \mathcal{S}_{\hat{a}}$, $\mathcal{S}^*_{\hat{a}}$ is sharper than $\mathcal{S}_{\hat{a}}$, but it still contains all true solutions of (P1). We have the following important observation.

Lemma 2 *The following inequalities hold*

$$\max_{(w,a)\in\mathcal{F}} V(w,a) \leq \max_{a} \inf_{\hat{a}} \max_{w} \{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}}\} \leq \max_{a} \inf_{\hat{a}} \max_{w} \{V(w,a) : (w,a) \in \mathcal{S}_{\hat{a}}\}.$$

Proof. First, we show the first inequality. For every $(w,a) \in \mathcal{F} \cap \{(w,a) : U(w,a) \geq U^*\}$, (RIC) and (IR*) are satisfied and

$$U(w,a) \geq U(w,\hat{a})$$

for any $\hat{a} \in \mathbb{A}$. Then, as at the optimum (w^{**}, a^{**}), (IR*) is binding, for any $a \in a^{BR}(w)$ such that $U(w,a) \geq U^*$, we have

$$\inf_{\hat{a}} \max_{w} \{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}}\} \geq \max_{w \in \{w : a^{BR}(w) = a \text{ and } U(w,a) \geq U^*\}} V(w,a),$$

which implies

$$\max_a \inf_{\hat{a}} \max_w \{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}}\} \geq \max_a \max_{w \in \{w : a^{BR}(w) = a \text{ and } U(w,a) \geq U^*\}} V(w,a) = \max_{(w,a) \in \mathcal{F}} V(w,a).$$

For the second inequality, we note that for every $(a, \hat{a})$, the maximum value

$$\max_w \{V(w,a) : U(w,a) \geq U, \text{ (RIC) and (NJ}(\hat{a}))\}$$

is non-increasing in U. Therefore, for $U^* \geq \underline{U}$,

$$\inf_{\hat{a}} \max_w \{V(w,a) : U(w,a) \geq U^*, \text{ (RIC) and (NJ}(\hat{a}))\} \leq \inf_{\hat{a}} \max_w \{V(w,a) : U(w,a) \geq \underline{U}, \text{ (RIC) and (NJ}(\hat{a}))\}.$$

■

The next step is to show the other direction of the inequality so that the order of optimization between w and $\hat{a}$ is switchable, i.e.,

$$\max_{(w,a) \in \mathcal{F}} V(w,a) = \max_{(w,a)} \inf_{\hat{a}} \{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}}\} = \max_a \inf_{\hat{a}} \max_w \{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}}\}, \tag{7}$$

for some sophisticatedly chosen U^*. Thus, we can characterize the optimal contract by (2) based on problem $(\text{P}|a, \hat{a}; U)$.

3.3 A Three-step Procedure

To show the equivalence property (7) and find a solution (w^{**}, a^{**}) of (P1), we use a three-step procedure, in a backward manner. First, Lemma 3 shows that for any implementable action, the optimal contract implementing it and delivering the same utility to the agent is an AMH contract. This step solves $\hat{a}$ in terms of a and U, through minimizing $\mathcal{L}^*(a, \hat{a}; U)$ over $\hat{a}$. Second, Lemma 4 solves the optimal action given the agent's utility U^* at optimum. After these two steps, the only unknown parameter in solving (P1) is U^*. So the third step pins down U^* by Lemma 5.

3.3.1 Optimal Contract Given an Implementable Action

Suppose $\tilde{a}^*$ is an implementable action, and the agent may earn utility U under that implementation. The first question is that, for any such action, what is the characterization of the optimal contract $\tilde{w}^*$ implementing $\tilde{a}^*$ and delivers at least utility

$$U = U(\tilde{w}^*, \tilde{a}^*)$$

to the agent?[10] More precisely, let $(\tilde{w}^*, \tilde{a}^*)$ as be such that

$$V(\tilde{w}^*, \tilde{a}^*) = \max_w \{V(w, \tilde{a}^*) : U(w, \tilde{a}^*) \geq U, \ \tilde{a}^* \in a^{BR}(w)\}, \tag{8}$$

we use Lemma 3 to characterize $\tilde{w}^*$.

[10] We ignore the action(s) that the principal may not be able to find an optimal contract to implement it, without loss of generality.

Lemma 3 *Assume that $u(.)$ is increasing and concave and $v(.)$ is increasing and weakly concave. For any given implementable $\tilde{a}^*$, the optimal contract $\tilde{w}^*$ implementing $\tilde{a}^*$ and delivering utility $U = U(\tilde{w}^*, \tilde{a}^*)$ should satisfy*

$$V(\tilde{w}^*, \tilde{a}^*) = \min_{\hat{a}} \mathcal{L}^*(\tilde{a}^*, \hat{a}; U), \tag{9}$$

where $V(\tilde{w}^*, \tilde{a}^*)$ is specified by (8).

Proof. Step 1.

By Lemma 1, the value of problem (P$|a, \hat{a}; U$) is continuous in $(a, \hat{a})$ and

$$\min_{\hat{a}} \mathcal{L}^*(\tilde{a}^*, \hat{a}; U) = \inf_{\hat{a}} \max_{w} \{V(w, \tilde{a}^*) : U(w, \tilde{a}^*) \geq U, \ (RIC), \text{ and } (\text{NJ}(\hat{a})) \text{ given } a = \tilde{a}^*\},$$

where the infimum is attainable by the continuity. It suffices to show that

$$\inf_{\hat{a}} \max_{w} \{V(w, \tilde{a}^*) : U(w, \tilde{a}^*) \geq U, \ (RIC), \text{ and } (\text{NJ}(\hat{a})) \text{ given } a = \tilde{a}^*\} = V(\tilde{w}^*, \tilde{a}^*).$$

Step 2.

Let $\tilde{w}^*$ be specified by (8). Suppose there is a deviation $h(x) \geq 0$, with $h(\underline{x}) = h(\overline{x}) = 0$. We parameterize a contract by

$$\tilde{w} = \tilde{w}^* + zh,$$

for $z \in \mathbb{R}$. We want to show that $z = 0$ is the optimal choice, by which $\tilde{w}^*$ is characterized.

For convenience, denote

$$\xi^{RIC}(a) = \{a : (\tilde{w}, a) \text{ satisfy } (RIC)\}.$$

Clearly, $\tilde{a}^* \in \xi^{RIC}(\tilde{a}^*)$ and $\tilde{a}^*$ is implemented when $z = 0$.[11] Now, we consider the following problem:

$$\inf_{\hat{a}} \max_{z} \{V(z, \tilde{a}^*) : U(z, \tilde{a}^*) \geq U, z \in \xi^{RIC}(\tilde{a}^*), \text{ and } U(z, \tilde{a}^*) - U(z, \hat{a}) \geq 0\}, \tag{10}$$

where we use the short notations $V(z, a) = V(\tilde{w}^* + zh, a)$ and $U(z, a) = U(\tilde{w}^* + zh, a)$.

First, as $\tilde{a}^*$ is incentive compatible, by the same reasoning as in Lemma 2, we have the inequality

$$\inf_{\hat{a}} \max_{z} \{V(z, \tilde{a}^*) : U(z, \tilde{a}^*) \geq U, z \in \xi^{RIC}(\tilde{a}^*), \text{ and } U(z, \tilde{a}^*) - U(z, \hat{a}) \geq 0\} \geq V(\tilde{w}^*, \tilde{a}^*). \tag{11}$$

Second, let $\hat{a}^*$ be a solution of (10) and

$$z^* \in \arg\max_{z} \{V(z, \tilde{a}^*) : U(z, \tilde{a}^*) \geq U, z \in \xi^{RIC}(\tilde{a}^*), \text{ and } U(z, \tilde{a}^*) - U(z, \hat{a}^*) \geq 0\}.$$

[11] If $\tilde{a}^*$ is an interior point, $\tilde{a}^*$ is a solution to the equation $\xi^{RIC}(a) = 0$. However, $\xi^{RIC}(a) = 0$ may have many solutions. Some dimensions of $\tilde{a}^*$ can also be corner solutions. Possibly, for a given $\tilde{a}^*$, $\xi^{RIC}(\tilde{a}^*)$ is not a singleton.

If inequality (11) is not binding, by $v'(.) > 0$, then $z^* < 0$ as $\frac{\partial V(z,\tilde{a}^*)}{\partial z} < 0$, which is in conflict with $U(z, \tilde{a}^*) \geq U$; therefore, $z^* = 0$.

Step 3.

Note that $V_z(z^*, \tilde{a}^*) < 0$ and $U_z(z^*, \tilde{a}^*) > 0$. Thus, the maximizer $z^* = 0$ should satisfy the necessary condition

$$V_z(z^*, \tilde{a}^*) + \lambda[U_z(z^*, \tilde{a}^*) - U] + \mu \cdot U_{az}(z^*, a^*) + \delta[U_z(z^*, \tilde{a}^*) - U_z(z^*, \hat{a}^*)] = 0,$$

for some (λ, μ, δ) and all three constraints are also satisfied, where $U_z(z, \tilde{a}^*)$, $U_{az}(z, a)$, and $U_z(z, \tilde{a}^*) - U_z(z, \hat{a}^*)$ denote the partial derivatives. Therefore, we obtain the first-order condition

$$\int \left(-v'(\pi - \tilde{w}^*) + u'(\tilde{w}^*)[\lambda + \mu \cdot l_a(x, \tilde{a}^*) + \delta(1 - \frac{f(x, \hat{a}^*)}{f(x, \tilde{a}^*)})] \right) h(x) f(x, \tilde{a}^*) dx = 0.$$

Note that our analysis is valid for any deviation $h(x) \geq 0$, with $h(\underline{x}) = h(\overline{x}) = 0$. Then $\tilde{w}^*$ should satisfy the first-order condition

$$-v'(\pi - \tilde{w}^*) + u'(\tilde{w}^*)[\lambda + \mu \cdot l_a(x, \tilde{a}^*) + \delta(1 - \frac{f(x, \hat{a}^*)}{f(x, \tilde{a}^*)})] = 0 \text{ for almost every } x;$$

or $\tilde{w}^* = \underline{w}$ if $\lambda + \mu \cdot l_a(x, \tilde{a}^*) + \delta(1 - \frac{f(x,\hat{a}^*)}{f(x,\tilde{a}^*)}) < \frac{v'(\pi - \underline{w})}{u'(\underline{w})}$. Recall that there is unique w solve the above first-order condition as we have discussed in (2), then $\tilde{w}^*$ is uniquely determined. Note also that $\tilde{w}^*$ satisfies the three constraints:

$$\begin{aligned} \lambda &\geq 0, \, U(\tilde{w}^*, \tilde{a}^*) - U \geq 0, \, \lambda \left(U(\tilde{w}^*, \tilde{a}^*) - U^* \right) = 0; \\ 0 &= U_{a_i}(\tilde{w}^*, \tilde{a}^*) \text{ or } \mu_i = 0 \text{ if } U_{a_i}(\tilde{w}^*, \tilde{a}^*) \neq 0, \, i = 1, 2, ..., N; \text{ and} \\ \delta &\geq 0, \, U(\tilde{w}^*, \tilde{a}^*) - U(\tilde{w}^*, \hat{a}^*) \geq 0, \text{ and } \delta[U(\tilde{w}^*, \tilde{a}^*) - U(\tilde{w}^*, \hat{a}^*)] = 0. \end{aligned}$$

By Lemma 1,

$$(\lambda, \mu, \delta) = (\lambda^*(\tilde{a}^*, \hat{a}^*; U), \mu^*(\tilde{a}^*, \hat{a}^*; U), \delta^*(\tilde{a}^*, \hat{a}^*; U))$$

is the unique Lagrangian multiplier vector that satisfies the three constraints at $z^* = 0$. Therefore,

$$\tilde{w}^* = w^*(., \tilde{a}^*, \hat{a}^*; U),$$

which means that $\tilde{w}^*$ also solves the problem

$$\max_w \{V(w, \tilde{a}^*) : U(w, \tilde{a}^*) \geq U, \text{ (RIC), and (NJ}(\hat{a}^*)) \text{ given } a = \tilde{a}^*\}.$$

Finally, as $\tilde{a}^*$ is implemented by $w^*(., \tilde{a}^*, \hat{a}^*; U)$, based on the same argument in Lemma 2 again, $\hat{a}^*$ is chosen by the minimization problem

$$\inf_{\hat{a}} \max_w \{V(w, \tilde{a}^*) : U(w, \tilde{a}^*) \geq U, \text{ (RIC), and (NJ}(\hat{a})) \text{ given } a = \tilde{a}^*\}.$$

■

According to Lemma 3, for any implementable action $\tilde{a}^*$, the Pareto optimal contract implementing $\tilde{a}^*$ and delivering the agent's utility no less than U is $w^*(.,\tilde{a}^*,\hat{a}^*;U)$. The intuition is that, having promised an at least utility level U to the agent, the principal cannot obtain a higher utility level than

$$\min_{\hat{a}} V(w^*(.,\tilde{a}^*,\hat{a};U),\tilde{a}^*),$$

because the principal has to prevent the agent from deviating to any $\hat{a}\in\mathbb{A}$. The key fact our theory is built on is a "zero-sum" feature of wage payment, given the output and action.

3.3.2 The Choice of Optimal Implementable $\tilde{a}^*$

In the second step, we show that the solution a^{**} is the optimal implementable $\tilde{a}^*$, given that the agent's utility level is at least $U^*=U(w^{**},a^{**})$.

Lemma 4 *Assume that $u(.)$ is increasing and concave, $v(.)$ is increasing and weakly concave, and (P1) has a solution (w^{**},a^{**}). Then,*

$$\max_{(w,a)\in\mathcal{F}} V(w,a)=\max_a\left(\inf_{\hat{a}}\max_w\{V(w,a):(IR^*),\ (RIC),\ \text{and}\ NJ(\hat{a})\}\right)=\max_a\min_{\hat{a}}\mathcal{L}^*(a,\hat{a};U^*), \tag{12}$$

and

$$a^{**}\in\arg\max_a\left(\min_{\hat{a}}\mathcal{L}^*(a,\hat{a};U^*)\right)$$

is an optimal action implemented by $w^{**}=w^*(.,a^{**},\hat{a}^*;U^*)$ for some $\hat{a}^*\in\arg\min_{\hat{a}}\mathcal{L}^*(a^{**},\hat{a};U^*)$.

Proof. Step 1.

Let $A^*=a^{BR}(w^{**})$ denote the collection of all best repsonses to the optimal contract w^{**}. Based on the theorem of maximum, A^* is compact. As the IC constraint is binding for any $a\in A^*$,

$$(w^{**},a^{**})\in\mathcal{S}^{**}\equiv\bigcap_{\hat{a}\in A^*}\mathcal{S}^*_{\hat{a}}.$$

If

$$\inf_{\hat{a}}\max_{(w,a)}\{V(w,a):(w,a)\in\mathcal{S}^*_{\hat{a}}\}=\max_{(w,a)\in\mathcal{F}}V(w,a),$$

we are done, by the minmax inequality

$$\max_a\left(\inf_{\hat{a}}\max_w\{V(w,a):(w,a)\in\mathcal{S}^*_{\hat{a}}\}\right)\le\inf_{\hat{a}}\max_{(w,a)}\{V(w,a):(w,a)\in\mathcal{S}^*_{\hat{a}}\}. \tag{13}$$

If

$$\inf_{\hat{a}}\max_{(w,a)}\{V(w,a):(w,a)\in\mathcal{S}^*_{\hat{a}}\}>\max_{(w,a)\in\mathcal{F}}V(w,a),$$

let $\hat{a}^* \in \mathbb{A}$ solve

$$\max_{(w,a)}\{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}^*}\} = \inf_{\hat{a}} \max_{(w,a)}\{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}}\}.$$

Note that

$$\max_{(w,a)\in\mathcal{F}} V(w,a) = \max_{(w,a)}\{V(w,a) : (w,a) \in \bigcap_{\hat{a}\in\mathbb{A}} \mathcal{S}^*_{\hat{a}}\},$$

then there must exist $(w^\#, a^\#) \in \mathcal{S}^*_{\hat{a}^*}$ but $(w^\#, a^\#) \notin \bigcap_{\hat{a}\in\mathbb{A}} \mathcal{S}^*_{\hat{a}}$ such that $V(w^\#, a^\#) > V(w^{**}, a^{**})$. Therefore, for any such $(w^\#, a^\#)$, we can choose some $\hat{a}^{**} \in \mathbb{A} - \{\hat{a}^*\}$ so that

$$\max_{w}\{V(w,a^\#) : (w,a^\#) \in \mathcal{S}^*_{\hat{a}^{**}}\} \leq \max_{w}\{V(w,a^\#) : (w,a^\#) \in \bigcap_{\hat{a}\in\mathbb{A}} \mathcal{S}^*_{\hat{a}}\},$$

where we define the maximum over an empty set $-\infty$.

As a result, any solution

$$\tilde{a}^* \in \arg\max_{a} \left(\inf_{\hat{a}} \max_{w}\{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}}\} \right)$$

must be chosen such that

$$\tilde{a}^* \in \{a : (w,a) \in \bigcap_{\hat{a}\in\mathbb{A}} \mathcal{S}^*_{\hat{a}}\} \subset \{a : (w,a) \in \bigcap_{\hat{a}\in A^*} \mathcal{S}^*_{\hat{a}}\},$$

which implies that $\tilde{a}^*$ should be implementable.

Step 2.

For every implementable $\tilde{a}^*$, by Lemma 3, we can find the optimal contract that induces action $\tilde{a}^*$ through (9). Therefore, for the solution $(\tilde{w}^*, \tilde{a}^*)$ such that

$$V(\tilde{w}^*, \tilde{a}^*) = \max_{a} \left(\inf_{\hat{a}} \max_{w}\{V(w,a) : (\text{IR}^*), (\text{RIC}), \text{ and } (\text{NJ}(\hat{a}))\} \right),$$

we have $(\tilde{w}^*, \tilde{a}^*) \in \mathcal{F}$. As a result

$$V(\tilde{w}^*, \tilde{a}^*) \leq \max_{(w,a)\in\mathcal{F}} V(w,a).$$

By Lemma 2, the other direction $V(\tilde{w}^*, \tilde{a}^*) \geq \max_{(w,a)\in\mathcal{F}} V(w,a)$ should hold. We have the desired conclusion $V(\tilde{w}^*, \tilde{a}^*) = \max_{(w,a)\in\mathcal{F}} V(w,a)$. Finally, $V(\tilde{w}^*, \tilde{a}^*) = \max_a \min_{\hat{a}} \mathcal{L}^*(a, \hat{a}; U^*)$ by Lemma 1. ■

Lemma 4 plays a key role in our approach, so it deserves further interpretations (see Figure 1). We use $\hat{a}$ to partition the relaxed feasible set $\mathcal{F}^R$ (Box) through the NJ constraints (circles). Each $\mathcal{S}^*_{\hat{a}}$ is a subset of $\mathcal{F}^R$. An $\hat{a}^{i*}$ solving (not necessarily unique)

$$\max_{(w,a)}\{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}^{i*}}\} = \inf_{\hat{a}} \left(\max_{(w,a)}\{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}}\} \right) \ (i = 1, 2, \ldots)$$

selects the single subset $\mathcal{S}^*_{\hat{a}^{i*}}$ (solid circle) that generates the lowest value among all $\mathcal{S}^*_{\hat{a}}$'s. However, there may exist some $(w^{\#}, a^{\#}) \in \mathcal{S}^*_{\hat{a}^{i*}}$ that is not feasible. The joint set $\underset{\hat{a}\in\mathbb{A}}{\cap} \mathcal{S}^*_{\hat{a}}$ (triangle area) is a subset of the original feasible constraint $\mathcal{F}$ (shaded circle) because of

$$\mathcal{F} = \left(\underset{\hat{a}\in\mathbb{A}}{\cap} \mathcal{S}_{\hat{a}}\right) \supset \left(\underset{\hat{a}\in\mathbb{A}}{\cap} \mathcal{S}^*_{\hat{a}}\right).$$

The problem

$$\arg\max_{a} \inf_{\hat{a}} \left(\max_{w}\{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}}\}\right)$$

further refines the choice set of a so that (w,a) belongs to $\underset{\hat{a}\in\mathbb{A}}{\cap} \mathcal{S}^*_{\hat{a}}$, therefore, the solution

$$\tilde{a}^* \in \arg\max_{a} \left\{\inf_{\hat{a}} \left(\max_{w}\{V(w,a) : (w,a) \in \mathcal{S}^*_{\hat{a}}\}\right)\right\}$$

must be implementable. Once $\tilde{a}^*$ is implementable, Lemma 3 shows that $w^*(., \tilde{a}^*, \hat{a}^*; U^*)$ is an optimal contract implementing $\tilde{a}^*$ and delivering "equilibrium" utility U^* to the agent. As a result $(w^*(., \tilde{a}^*, \hat{a}^*; U^*), \tilde{a}^*) \in \mathcal{F}$, we obtain the other direction of the first inequality in Lemma 2. Clearly, a^{**} is the best implementable $\tilde{a}^*$ (the core inside the shadow circle).

(Insert Figure 1 here)

In some special cases, even if we do not know U^* yet (so that the optimal w^{**} may be unknown), we may find a^{**} or some $\tilde{a}^* \in A^*$ based on the following corollary.

Corollary 1 *Assume that the conditions in Lemma 4 hold. Then, the solution*

$$\tilde{a}^* \in \arg\max_{a} \left(\min_{\hat{a}} \mathcal{L}^*(a, \hat{a}; \underline{U})\right)$$

must be chosen such that

$$\tilde{a}^* \in \{a : (w,a) \in \underset{\hat{a}\in A^*}{\cap} \mathcal{S}_{\hat{a}}\} \text{ for } A^* = a^{BR}(w^{**}).$$

Proof. In Step 1 of the Proof of Lemma 4, if we replace $\mathcal{S}^*_{\hat{a}}$ by $\mathcal{S}_{\hat{a}}$, then all reasonings go through. For any $\tilde{a}^* \notin \{a : (w,a) \in \underset{\hat{a}\in A^*}{\cap} \mathcal{S}_{\hat{a}}\}$, there exists an $\hat{a}^*$ such that we can exclude $\tilde{a}^*$ from set $\mathcal{S}_{\hat{a}^*}$. ■

3.3.3 The Agent's Utility at the Optimum

By Lemmas 3 and 4, everything now boils down to a real number U^*. Once we determine U^*, the original problem (P1) can be solved by equivalence (12). Of course, in many interesting situations, the IR constrait is binding at the optimum, i.e., $U^* = \underline{U}$. Therefore, by Lemmas 3 and 4, the problem (P1) is solved.

Proposition 1 *Assumes that the conditions in Lemma 4 are satisfied. Then, the equivalence property (7) holds when IR is binding at the optimum. In particular, when a is one-dimensional, the IR constraint is binding (see Appendix A2 for the proof).*

If at the optimum, the IR constraint is not binding, then U^* is an unknown number, which can be determined as follows.

Lemma 5 *Assume that the conditions in Lemma 4 are satisfied. Then, the agent's utility level at the optimum is*

$$U^* = \min\left\{\arg\min_{U\geq \underline{U}}\left(\mathcal{L}^*(a^*(U),\hat{a}^*(U);U) - \max_{a\in a^{BR}(w^*(.,a^*(U),\hat{a}^*(U);U))} V(w^*(.,a^*(U),\hat{a}^*(U);U),a)\right)\right\}, \tag{14}$$

where $(a^*(U),\hat{a}^*(U)$ *solves*

$$\mathcal{L}^*(a^*(U),\hat{a}^*(U);U) = \max_a\left(\min_{\hat{a}}\mathcal{L}^*(a,\hat{a};U)\right).$$

Proof. For any $U^* \geq U \geq \underline{U}$, we have

$$\begin{aligned}\mathcal{L}^*(a^*(U),\hat{a}^*(U);U) &\geq \mathcal{L}^*(a^*(U^*),\hat{a}^*(U^*);U^*) = \max\{V(w,a) : a\in a^{BR}(w) \text{ and } U(w,a)\geq U^*\}\\ &\geq \max_{a\in a^{BR}(w^*(.,a^*(U),\hat{a}^*(U);U))} V(w^*(.,a^*(U),\hat{a}^*(U);U),a),\end{aligned}$$

where the first inequality is due to the fact that $V^*(U)$ is non-increasing in U, the second inequality is by Lemma 4, and the third inequality is because of

$$(w^*(.,a^*(U),\hat{a}^*(U);U), a^{BR}(w^*(.,a^*(U),\hat{a}^*(U);U)))\subset\mathcal{F}$$

for any $U\in[\underline{U},U^*]$. Therefore, U^* is a solution minimizing the gap

$$\mathcal{L}^*(a^*(U),\hat{a}^*(U);U) - \max_{a\in a^{BR}(w^*(.,a^*(U),\hat{a}^*(U);U))} V(w^*(.,a^*(U),\hat{a}^*(U);U),a).$$

We select the smallest $U\geq\underline{U}$ that solves

$$\mathcal{L}^*(a^*(U),\hat{a}^*(U);U) - \max_{a\in a^{BR}(w^*(.,a^*(U),\hat{a}^*(U);U))} V(w^*(.,a^*(U),\hat{a}^*(U);U),a) = 0,$$

which is the right U^* that gives the highest utility to the principal. ■

We provide some intuitions for Lemma 5. For any $U\in[\underline{U},U^*]$, the term $V^*(U)$ is the maximum value of a relaxed problem, which is greater than the maximum value being generated by any incentive compatible contract that delivers at least utility U to the agent by a similar argument in Lemma 2. Meanwhile, $w^*(.,a^*(U),\hat{a}^*(U);U)$ is a special contract that delivers at least utility U to the agent. Therefore, the true value of (P1) should lie in between these two terms. When we increase U from $\underline{U}$, the gap disappears until $U=U^*$. The smallest U that elicits the gap is the one we are looking for.

By Lemma 5, we close the problem. The result is summarized by the following main theorem.

Theorem 1 *Assume that $u(.)$ is increasing and concave, $v(.)$ is increasing and weakly concave, and (P1) has a solution. Thus,*

$$\max_{(w,a)\in\mathcal{F}} V(w,a) = \max_{a}\left(\inf_{\hat{a}}\max_{w}\{V(w,a) : U(w,a)\geq U^*,\ (RIC),\ and\ (NJ)\}\right) = \max_{a}\min_{\hat{a}}\mathcal{L}^*(a,\hat{a};U^*),$$

where U^* is determined by (14).

Proof. By Lemma 5, we find the agent's utility at the optimum. Fix that particular utility U^*, the optimal action a^{**} is solved by Lemma 4, and the corresponding contract $w^*(.,a^{**},\hat{a}^*;U^*)$ is solved by Lemma 3 for some $\hat{a}^*\in\arg\min_{\hat{a}}\mathcal{L}^*(a^{**},\hat{a};U)$. ■

Like Grossman and Hart (1983), Theorem 1 is a general solution method for the principal-agent problem. Moreover, Theorem 1 provides a parsimonious way to characterize the optimal contract, which is the first close form optimal contract without FOA, using only one no-jumping condition. Based on Theorem 1, we may learn some information about the optimal contract from equation (2). Equation (2) has a very clear economic interpretation. The left-hand side is usually regarded as the marginal incentive cost (Grossman and Hart, 1983). The right-hand side is a mean-preserving spread (MPS) of λ (the shadow price of IR constraint) because the means of both $l_a(x,a)$ and $(1-\frac{f(x,\hat{a})}{f(x,a)})$ are zero. Note that, in the case $\delta>0$, where the FOA is invalid, the marginal incentive cost $\frac{v'(\pi(x)-w_{\lambda,\mu,\delta}(x,a,\hat{a}))}{u'(w_{\lambda,\mu,\delta}(x,a,\hat{a}))}$ is an MPS of $\frac{v'(\pi(x)-w_{\lambda,\mu}(x,a))}{u'(w_{\lambda,\mu}(x,a))}$. So when the FOA is invalid, an extra dispersion is needed.[12] In Section 4, we will provide several applications based on our new characterization.

3.4 The Necessary Optimality Conditions

This subsection provides some necessary optimality conditions for a or $\hat{a}$ based on the Lagrangian method. First, to solve problem (P2), we may try the following problem

$$\inf_{\hat{a}}\max_{(w,a)}\{V(w,a) : U(w,a)\geq U,\ (RIC),\ \text{and}\ (\text{NJ}(\hat{a}))\}. \tag{P3}$$

The reason is that problem (P3) usually has a more tractable necessary optimality condition problem than (P2) does. The Kuhn-Tucker necessary conditions are applicable because $\mathcal{L}(w,a,\hat{a};\lambda,\mu,\delta;U)$ is continuous and differentiable in each argument.

Proposition 2 *Assume that $u(.)$ is increasing and concave and $v(.)$ is increasing and weakly concave. Then, given any $U\geq\underline{U}$, for any a^* and $\hat{a}^*$ such that $\mathcal{L}^*(a^*,\hat{a}^*;U)=\min_{\hat{a}}\max_a\mathcal{L}^*(a,\hat{a};U)$, we have: (i) if a^* is a local extremum of $U(w^*(.,a^*,\hat{a}^*;U),a)$ in a, for an interior solution $\hat{a}_i^*\in(\underline{a}_i,\bar{a}_i)$,*

$$\frac{\partial}{\partial\hat{a}_i}\mathcal{L}^*(a^*,\hat{a}^*;U) = -\delta^*(a^*,\hat{a}^*;U)U_{a_i}(w^*(.,a^*,\hat{a}^*;U),\hat{a}^*)=0,$$

[12] Recently, Kadan and Swinkels (2012) provide an elegant analysis about the general implication of $\mathbb{E}(\frac{1}{u'(w)})=\lambda$.

and $U_{a_i}(w^*(.,a^*,\hat{a}^*;U),\hat{a}^*) \geq 0$ *(*≤ 0*) for* $\hat{a}_i^* = \bar{a}_i$ *(*$\hat{a}_i^* = \underline{a}_i$*); (ii) for an interior solution* $a_i^* \in (\underline{a}_i, \bar{a}_i)$,

$$\frac{\partial}{\partial a_i}\mathcal{L}^*(a^*,\hat{a}^*;U) = V_{a_i}(w^*(.,a^*,\hat{a}^*;U),a^*) + \mu^*(a^*,\hat{a}^*;U)\cdot U_{aa_i}(w^*(.,a^*,\hat{a}^*;U),a^*) = 0;$$

for a corner solution, $a_i^* = \underline{a}_i$ *(*$a_i^* = \bar{a}_i$*)* $\frac{\partial}{\partial a_i}\mathcal{L}^*(a^*,\hat{a}^*;U) \leq 0$ *(*≥ 0*), where* $i = 1, 2, ..., N$.

Proof. For part (i), note that whenever $\delta^*(a^*,\hat{a}^*;U) > 0$, $\hat{a}^*$ is also a local extremum of $U(w^*(.,a^*,\hat{a}^*;U),\hat{a})$ in $\hat{a}$. When $\delta^*(a^*,\hat{a}^*;U) = 0$, $\hat{a}^*$ does not appear in w^*. Part (ii) is based on that the Lagrangian dual $\mathcal{L}(w_{\lambda,\mu,\delta},a,\hat{a};\lambda,\mu,\delta;U)$ is strictly convex in (λ,μ,δ) for every $(a,\hat{a})$ (see Appendix A1 for the detail). Therefore,

$$\min_{(\lambda,\mu,\delta)} \mathcal{L}(w_{\lambda,\mu,\delta},a,\hat{a};\lambda,\mu,\delta;U)$$

is differentiable and by the envelope theorem, the first-order condition $\frac{\partial}{\partial a_i}\mathcal{L}^*(a^*,\hat{a}^*;U) = 0$ holds for the interior point of a_i^* under contract $w^*(.,a^*,\hat{a}^*;U)$. ■

The necessary optimality conditions provided by Proposition 2 are useful. We only need to consider a small set of $(a,\hat{a})$ with a known functional form $w_{\lambda,\mu,\delta}(x,a,\hat{a})$.

If (P3) is not equivalent to the original problem, which means the minmax inequality (13) is not binding

$$\min_{\hat{a}}\max_{a}\mathcal{L}^*(a,\hat{a};U) > \max_{a}\min_{\hat{a}}\mathcal{L}^*(a,\hat{a};U),$$

we can move forward to solve problem $\max_a \min_{\hat{a}} \mathcal{L}^*(a,\hat{a};U)$.

Proposition 3 *Assume that* $u(.)$ *is increasing and concave and* $v(.)$ *is increasing and weakly concave. Given any* $U \geq \underline{U}$*, for any* a^* *and* $\hat{a}^*$ *such that* $\mathcal{L}^*(a^*,\hat{a}^*;U) = \max_a \min_{\hat{a}} \mathcal{L}^*(a,\hat{a};U)$*, we have (i) for an interior solution* $\hat{a}_i^* \in (\underline{a}_i, \bar{a}_i)$,

$$\frac{\partial}{\partial \hat{a}_i}\mathcal{L}^*(a^*,\hat{a}^*;U) = -\delta^*(a^*,\hat{a}^*;U)U_{a_i}(w^*(.,a^*,\hat{a}^*;U),\hat{a}^*) = 0,$$

and $U_{a_i}(w^*(.,a^*,\hat{a}^*;U),\hat{a}^*) \geq 0$ *(*≤ 0*) for* $\hat{a}_i^* = \bar{a}_i$ *(*$\hat{a}_i^* = \underline{a}_i$*); (ii) for an interior solution* $a_i^* \in (\underline{a}_i, \bar{a}_i)$*, the right derivative*

$$\frac{\partial}{\partial a_i^+}\left(\min_{\hat{a}}\mathcal{L}^*(a^*,\hat{a};U)\right) \leq 0,$$

and left derivative

$$\frac{\partial}{\partial a_i^-}\left(\min_{\hat{a}}\mathcal{L}^*(a^*,\hat{a};U)\right) \geq 0;$$

and (iii) for a corner solution, $a_i^* = \underline{a}_i$ *(*$a_i^* = \bar{a}_i$*)* $\frac{\partial}{\partial a_i}(\min_{\hat{a}}\mathcal{L}^*(a^*,\hat{a};U)) \leq 0$ *(*≥ 0*).*

Proof. For part (i), as

$$\min_{\hat{a}} \min_{\lambda,\mu,\delta} \max_{w} \mathcal{L}(w, a, \hat{a}; \lambda, \mu, \delta; U) = \min_{\lambda,\mu,\delta} \min_{\hat{a}} \max_{w} \mathcal{L}(w, a, \hat{a}; \lambda, \mu, \delta; U),$$

the desired result follows from the envelope theorem. For part (ii), note that $\min_{\hat{a}} \mathcal{L}^*(a^*, \hat{a}; U)$ is continuous and directionally differentiable in a (see e.g., Corollary 4.4, Dempe, 2002). Because a^* is a maximum, the sign of $\frac{\partial}{\partial a_i}(\min_{\hat{a}} \mathcal{L}^*(a^*, \hat{a}; U))$ should go the opposite way (weakly) in the two different sides of a_i^*. ■

Corollary 2 *Assume that $u(.)$ is increasing and concave, $v(.)$ is increasing and weakly concave and (P1) has a solution. Then, whenever the IR constraint is binding at the optimum, we have*

$$\max_{(w,a)\in\mathcal{F}} V(w, a) = \max_{a} \min_{\hat{a}} \mathcal{L}^*(a, \hat{a}; \underline{U}) = \max_{(w,a)} \min_{(\lambda,\mu,\delta)} \min_{\hat{a}} \mathcal{L}(w, a, \hat{a}; \lambda, \mu, \delta; \underline{U}).$$

Proof. By Theorem 1, $V(w^{**}, a^{**}) = \max_a \min_{\hat{a}} \mathcal{L}^*(a, \hat{a}, \underline{U})$ when the IR constraint is binding at the optimum. Moreover, by the minmax inequality, we have

$$\begin{aligned} & V(w^{**}, a^{**}) \\ = \; & \max_{a} \min_{\hat{a}} \mathcal{L}^*(a, \hat{a}; \underline{U}) \geq \min_{(\lambda,\mu,\delta)} \max_{(w,a)} \min_{\hat{a}} \mathcal{L}(w, a, \hat{a}; \lambda, \mu, \delta; \underline{U}) \geq \max_{(w,a)} \min_{(\lambda,\mu,\delta)} \min_{\hat{a}} \mathcal{L}(w, a, \hat{a}; \lambda, \mu, \delta; \underline{U}) \end{aligned}$$

Note that $\arg\min_{\hat{a}} \mathcal{L}(w, a, \hat{a}; \lambda, \mu, \delta; \underline{U})$ will select a best response $\hat{a} \in a^{BR}(w)$, therefore,

$$\max_{(w,a)} \min_{(\lambda,\mu,\delta)} \min_{\hat{a}} \mathcal{L}(w, a, \hat{a}; \lambda, \mu, \delta; \underline{U}) \geq \min_{(\lambda,\mu,\delta)} \min_{\hat{a}} \mathcal{L}(w^{**}, a^{**}, \hat{a}; \lambda, \mu, \delta; \underline{U}) \geq V(w^{**}, a^{**}),$$

where the first inequality is by maximum property, and the last step is due to $U(w^{**}, a^{**}) \geq U(w^{**}, \hat{a})$ for any $\hat{a}$ by $a^{**} \in a^{BR}(w^{**})$. The desired conclusion follows. ■

Corollary 2 can help in terms of changing the order optimization of problem (P3). In some cases, conducting optimization over $\hat{a}$ first may be more plausible.

Remark 1 *The analysis in this section can apply to the case where x is discrete, $f(x, a)$ is a probability mass function, and the integral is replaced by a Lebesgue integral.*

Remark 2 *The technique developed here can be applied to a general bi-level parametric optimization problem such as*

$$\max_{x,y}\{F(x, y) : G(x, y) \geq 0, \; y \in \omega(x)\},$$

where

$$\omega(x) \equiv \arg\max_{y}\{f(x, y) : g(x, y) \geq 0\}$$

denotes the set of solutions of the agent's problem. A large body of literature on bilevel programming (see, e.g., Dempe, 2002; Colson et al., 2007). The advantage of our main result is that determining the number of best responses and their topological structure is not required, which is usually indispensable in the existing bilevel programming literature.

We provide an example from Araujo and Moreira (2001) to illustrate the advantage of the current approach when the FOA is invalid. They used an algorithm to compute 20 non-linear systems based on Kuhn-Tucker's conditions and the first-order conditions, whereas we used a straightforward calculation in our approach. The example shows that the corner solution constraint is automatically implied by the no-jumping constraint.

Example 1 *(Araujo and Moreira, 2001). The principal has expected utility* $V(w,a) = \sum_{i=1}^{2} p_i(a)(x_i - w_i)$*, where* $p_1(a) = 1 - a^3$*,* $p_2(a) = a^3$ *for* $a \in [0, 0.9]$*, and* $x_1 = 1$*,* $x_2 = 5$*, and the agent's expected utility is* $U(w,a) = \sum_{i=1}^{2} p_i(a)\sqrt{w_i} - a^2$ *with reservation utility* $\underline{U} = 0$*. It is easy to check that the FOA is invalid. Formula (3) in this case becomes*

$$w_{\lambda,\mu,\delta}(x_i, a, \hat{a}) = \frac{1}{4}(\lambda + \mu\frac{p_i'(a)}{p_i(a)} + \delta[1 - \frac{p_i(\hat{a})}{p_i(a)}])^2.$$

According to the procedure in Lemma 1,

$$\lambda^*(a,\hat{a}) = 2a^2, \mu^*(a,\hat{a}) = 0, \text{ and } \delta^*(a,\hat{a}) = \frac{2a^3(a+\hat{a})(1-a^3)}{(a-\hat{a})(a^2+a\hat{a}+\hat{a}^2)^2},$$

and

$$\begin{aligned}\mathcal{L}^*(a,\hat{a}) &= \sum_{i=1}^{2} p_i(a)(x_i - w_{\lambda^*,0,\delta^*}(x_i, a, \hat{a})) \\ &= (1-a^3) + 5a^3 - \frac{a^3(a\hat{a}^2(2a^2+2a\hat{a}+\hat{a}^2) + (a+\hat{a})^2)}{(a^2+a\hat{a}+\hat{a}^2)^2}.\end{aligned}$$

As $\frac{\partial}{\partial \hat{a}}\mathcal{L}^*(a,\hat{a}) > 0$*,* $\hat{a}^* = 0$ *by minimizing* $\mathcal{L}^*(a,\hat{a})$ *over* $\hat{a}$*. Then,*

$$\mathcal{L}^*(a,0) = (1-a^3) + 5a^3 - a$$

has a maximium at $a^* = 0.9$*. We can check whether* $a^* = 0.9$ *is implementable as follows.*

The agent's expected utility under $w_{\lambda^*,0,\delta^*}$ *is*

$$U(w_{\lambda^*,0,\delta^*}, \tilde{a}) = \frac{(a-\tilde{a})(\hat{a}-\tilde{a})(a\hat{a}+\tilde{a}(a+\hat{a}))}{(a^2+a\hat{a}+\hat{a}^2)}.$$

When $\tilde{a} > \frac{2(a^2+a\hat{a}+\hat{a}^2)}{3(a+\hat{a})}$*,* $U(w_{\lambda^*,0,\delta^*}, \tilde{a})$ *increases in* $\tilde{a}$*, and when* $\tilde{a} < \frac{2(a^2+a\hat{a}+\hat{a}^2)}{3(a+\hat{a})}$*,* $U(w_{\lambda^*,0,\delta^*}, \tilde{a})$ *decreases in* $\tilde{a}$*. Therefore, the best response must be in the corner(s). When* $\hat{a} = 0$ *and* $a = 0.9$*, the best response is* $a^* = 0.9 \in a^{BR}(w^*(., a^*, \hat{a}^*))$*, which is a fixed point.*

4 Applications

In this section, we apply our characterization of an optimal contract to rank the efficiency of signals and investigate the curvature of the optimal contract.

4.1 Ranking of Signals without the Validity of the FOA

Holmstrom (1979) first proves that an additional signal improves the efficiency if and only if it changes the likelihood ratio, which is known as the *sufficient statistic* criterion. Since in applications information may not be inclusive, Kim (1995) generalizes Holmstrom's criterion to the mean-preserving spread (MPS) criterion, which can rank the signals in non-inclusive environments. The MPS criterion is shown to be necessary and sufficient for ranking signals in a single-task case by Jewitt (2007). Recently, Xie (2011) generalizes the Kim's MPS criterion to the lift zonoid criterion for the multi-task case. These criteria are based on the distribution function alone and deliver very neat intuition for understanding the issue of asymmetric information in an agency model.

However, all of these results are proved based on a strong assumption that the agent's expected utility $U(w, a)$ is globally concave in a or the FOA is valid, which significantly shrinks the range of applicability of these criteria. For example, suppose X and Y are two independent random variables with an exponential distribution, respectively. Ignoring the issue of the FOA, information system X and (X, Y) are comparable according to Holmstrom's criterion, where for a bit abuse of notation, capital case X or Y also denotes an information system corresponding to the random variable X or Y. However, the exponential distribution only satisfies the monotone likelihood ratio property (MLRP) but not the convexity distribution function condition (CDFC) so that the FOA may be invalid. Which one is more efficient is unclear. In fact, assuming the validity of the FOA may exclude many interesting distributions since many commonly used distributions fail CDFC and MLRP. When the signal or task is multi-dimensional, the sufficient condition for the validity of the FOA is even more demanding. We now use Theorem 1 and the AMH contract implied by (2) to provide a theory for ranking the information system without the FOA. We will show that the sufficient statistic criterion remains valid and modify the MPS criterion to obtain several new criteria.

Following Kim (1995), we consider a risk-neutral principal. For any implementable a, the principal's profit maximization is equivalent to the compensation cost

$$\min_{w \in \{w:(w,a)\in\mathcal{F}\}} \int wf(x, a)dx.$$

We say information system X is *more efficient at a* than Y, if and only if the expected compensation cost for implementing a under X is lower than that under Y. If this property holds weakly for all a, and strictly at least for some a, then we say X is more efficient than Y.

With a bit abuse of notation, let U^* and U^*_a denote the agent's utility and marginal utility vector under an optimal contract implementing a. Therefore, by Theorem 1, we have the following equivalence

$$\begin{aligned}&\min_{w\in\{w:(w,a)\in\mathcal{F}\}}\int wf(x,a)dx\\&=\max_{\hat{a}}\min_w\{\int wf(x,a)dx:\int u(w)f(x,a)dx-c(a)\geq U^*,\ \int u(w)f_a(x,a)dx-c_a(a)=U^*_a,\\&\int u(w)f(x,a)dx-c(a)\geq\int u(w)f(x,\hat{a})dx-c(\hat{a})\}\\&=\max_{\hat{a}}\max_{\lambda,\mu,\delta}\min_w\mathcal{L}^c(w,a,\hat{a};\lambda,\mu,\delta).\end{aligned}$$

where $\mathcal{L}^c(.)$ is the Lagrangian for the cost minimization problem

$$\begin{aligned}&\mathcal{L}^c(w,a,\hat{a};\lambda,\mu,\delta)=\int[w-(\lambda+\mu\cdot l_a(x,a)+\delta(1-\tfrac{f(x,\hat{a})}{f(x,a)}))u(w)]f(x,a)dx\\&+\lambda(c(a)+U^*)+\mu\cdot(U^*_a+c_a(a))+\delta(c(a)-c(\hat{a})).\end{aligned}$$

We have the following result.

Proposition 4 *(Generalized MPS criterion) Assume that $u(.)$ is increasing and strictly concave, the principal is risk neutral, and (P1) has a solution. For two signals X and Y, X is more efficient than Y for a given implementable $a\notin\arg\min_{a'\in\mathbb{A}}c(a')$ if and only if the random variable*

$$\mu\cdot l_a(X,a)+\delta(1-\frac{f(X,\hat{a})}{f(X,a)})\text{ is a MPS of }\mu\cdot l_a(Y,a)+\delta(1-\frac{f(Y,\hat{a})}{f(Y,a)})\text{ for any }(\mu,\delta,\hat{a}).\quad\text{(G-MPS)}$$

Proof. Sufficiency. Let $(\lambda_I,\mu_I,\delta_I,\hat{a}_I)$ $(I=X,Y)$ be the maximizer of the dual $\min_w\mathcal{L}^c(w,a,\hat{a};\lambda,\mu,\delta)$ of information sysmtem I, given a. To show that X is more efficient than Y, it suffices to show

$$\begin{aligned}&\min_w\mathcal{L}^c(w,a,\hat{a}_Y;\lambda_Y,\mu_Y,\delta_Y)\\&\geq\Big(\min_w\int[w-(\lambda_X+\mu_X\cdot l_a(y,a)+\delta_X(1-\tfrac{f(y,\hat{a}_X)}{f(y,a)}))u(w)]f(y,a)dy\Big)\\&+\lambda_X(c(a)+U^*)+\mu_X\cdot(U^*_a+c_a(a))+\delta_X(c(a)-c(\hat{a}_X))\\&>\Big(\min_w\int[w-(\lambda_X+\mu_X\cdot l_a(x,a)+\delta_X(1-\tfrac{f(x,\hat{a}_X)}{f(x,a)}))u(w)]f(x,a)dx\Big)\\&+\lambda_X(c(a)+U^*)+\mu_X\cdot(U^*_a+c_a(a))+\delta_X(c(a)-c(\hat{a}_X))\\&=\min_w\mathcal{L}^c(w,a,\hat{a}_X;\lambda_X,\mu_X,\delta_X),\end{aligned}$$

where the first and last step are by the definition of maximum and the key step is thus to show the second inequality. Following the classical argument by Kim (1995), let

$$q(.)=\lambda_X+\mu_X\cdot l_a(.,a)+\delta_X(1-\frac{f(.,\hat{a}_X)}{f(.,a)}).$$

Note that $\min_w[w-qu(w)]$ is concave in q, so

$$\int\min_w[w-q(y)u(w)]f(y,a)dy\geq\int\min_w[w-q(x)u(w)]f(x,a)dx\tag{15}$$

if and only if $q(X)$ is a MPS of $q(Y)$. Canceling the λ_X in both sides, it is equivalent to condition (G-MPS), which implies inequality (15). And the inequality is strict unless the limited liability constraint $w \geq \underline{w}$ binds with probability 1, in which the agent will choose $a \in \arg\min_{a' \in \mathbb{A}} c(a')$ so the two systems are the same. We obtain the desired conclusion.

Necessity. The idea follows from Jewitt (2007). By contradiction, suppose for a set of $(\lambda, \mu, \delta, \hat{a})$, (G-MPS) condition fails, then under the parameters $(\lambda, \mu, \delta, \hat{a})$, there exist some utility $u(.)$ such that

$$\begin{aligned} & \min_w \int [w - (\lambda + \mu \cdot l_a(x,a) + \delta(1 - \frac{f(x,\hat{a})}{f(x,a)}))u(w)] f(x,a)dx \\ > & \min_w \int [w - (\lambda + \mu \cdot l_a(y,a) + \delta(1 - \frac{f(y,\hat{a})}{f(y,a)}))u(w)] f(y,a)dy. \end{aligned}$$

We have

$$\begin{aligned} & \min_w \mathcal{L}^c(w, a, \hat{a}_X; \lambda_X, \mu_X, \delta_X) \\ \geq & \min_w \mathcal{L}^c(w, a, \hat{a}; \lambda, \mu, \delta) \\ > & \left(\min_w \int [w - (\lambda + \mu \cdot l_a(y,a) + \delta(1 - \frac{f(y,\hat{a})}{f(y,a)}))u(w)] f(y,a)dy \right) \\ & + \lambda(c(a) + U^*) + \mu \cdot (U_a^* + c_a(a)) + \delta(c(a) - c(\hat{a})) \\ = & \int u(\max\{\underline{w}, u'^{-1}(\frac{1}{\lambda + \mu \cdot l_a(y,a) + \delta(1 - \frac{f(y,\hat{a})}{f(y,a)})})\}) f(y,a)dy, \end{aligned}$$

where the last step is because there always exists a convex cost function $c(a)$ such that the three constraints

$$\begin{cases} \int u(w) f(y,a)dx - c(a) = U^* \\ \int u(w) f_a(y,a)dx - c_a(a) = U_a^* \\ \int u(w) f(y,a)dx - c(a) = \int u(w) f(y,\hat{a})dy - c(\hat{a}) \end{cases}$$

are satisfied when $w = \max\{\underline{w}, u'^{-1}(\frac{1}{\lambda+\mu \cdot l_a(y,a)+\delta(1-\frac{f(y,\hat{a})}{f(y,a)})})\}$. For example, we can choose $c(a) = k_0 + \sum_{i=1}^N k_i a_i + \sum_{j=1}^N \sum_{i=1}^N a_i c_{ij} a_j$ to have enough parameters to meet the three constraints for any $(\lambda, \mu, \delta, \hat{a})$. Therefore,

$$\int u(\max\{\underline{w}, u'^{-1}(\frac{1}{\lambda + \mu \cdot l_a(y,a) + \delta(1 - \frac{f(y,\hat{a})}{f(y,a)})})\}) f(y,a)dy \geq \min_w \mathcal{L}^c(w, a, \hat{a}_Y; \lambda_Y, \mu_Y, \delta_Y),$$

because $\max\{\underline{w}, u'^{-1}(\frac{1}{\lambda+\mu \cdot l_a(y,a)+\delta(1-\frac{f(y,\hat{a})}{f(y,a)})})\}$ satisfies the three constraints but may not be a minimizer of the constrained minimization problem. This is a contradiction to that X is more efficient than Y. ■

Remark 3 *The sufficiency is true for any cost function $c(.)$, but the necessity holds even if $c(.)$ is monotone and convex.*

Clearly, since $\hat{a}$ is arbitrary, the necessary conditions with assumption of the FOA are still necessary for our generalized MPS criterion (G-MPS). However, for the sufficiency, we have one aditional term $(1 - \frac{f(.,\hat{a})}{f(.,a)})$, scaled by a non-negative number, which comes from the no-jumping constraint. This term is a global likelihood ratio, while the term $l_a(.,a)$ is regarded as a local likelihood ratio when $\hat{a} \to a$. The new lesson we learn is that, the information based on local likelihood ratio is not enough for ranking, when the FOA is invalid.[13] We do need some global information of likelihood ratio, because the agent's actions may jump dramatically without the FOA. We are wondering if any existing criterion remains valid according to Proposition 4. The following proposition shows that Holmstrom's criterion is robust.

Proposition 5 *Assume that conditions in Proposition 4 hold. Then, information system (X, Y) is more efficient than X if and only if X is not a sufficient statistic for (X, Y).*

Proof. Sufficiency. By contradiction, suppose X is a sufficient statistic of (X, Y). Then, the distribution of (X, Y) can be written as

$$f(x, y, a) = \phi(x, y) f(x, a),$$

for every (x, y) and $a \in \mathbb{A}$. Therefore, $l_a(x, y, a) = l_a(x, a)$ and $\frac{f(x,y,\hat{a})}{f(x,y,a)} = \frac{f(x,\hat{a})}{f(x,a)}$. The conclusion follows.

Necessity. Let $\phi(y\,|x, a)$ be conditional distribution of Y given X, we decompose

$$f(x, y, a) = \phi(y\,|x, a) f(x, a).$$

Therefore, by some algebra, the generalized MPS criterion (G-MPS) implies that

$$\mathbb{E}\left[\mu \cdot \frac{\phi_a(Y\,|X, a)}{\phi(Y\,|X, a)} + \delta\frac{f(X, \hat{a})}{f(X, a)}(1 - \frac{\phi(Y\,|X, \hat{a})}{\phi(Y\,|X, a)}) \middle| \mu \cdot l_a(X, a) + \delta(1 - \frac{f(X, \hat{a})}{f(X, a)})\right] = 0$$

for any $(\mu, \delta, \hat{a})$. Thus, for any vector $\alpha \in \mathbb{R}^N$ and $\tau \in \mathbb{A}$,

$$\mathbb{E}[\alpha \cdot \frac{\phi_a(Y\,|X, \tau)}{\phi(Y\,|X, \tau)}\,|\alpha \cdot l_a(X, \tau)] = 0,$$

Choosing $\alpha = \hat{a} - a$ and $a \le \tau \le \hat{a}$ such that $\log\phi(Y\,|X, \hat{a}) - \log\phi(Y\,|X, a) = (\hat{a} - a) \cdot l_a(X, \tau)$, we have

$$\mathbb{E}[\log\frac{\phi(Y\,|X, \hat{a})}{\phi(Y\,|X, a)}\left|\log\frac{f(X, \hat{a})}{f(X, a)}\right.] = \mathbb{E}[\log\frac{\phi(Y\,|X, \hat{a})}{\phi(Y\,|X, a)}\left|\frac{f(X, \hat{a})}{f(X, a)}\right.] = 0.$$

[13] With the FOA, we may compare two systems for a given a based on the local likelihood ratio $l_a(x, a)$ at given a. Without the FOA, we need the global information about $l_a(x, a')$ for $a' \in \mathbb{A}$ even if we only conduct compare for a specific action a.

Meanwhile, from condition (G-MPS),

$$\mathbb{E}[\frac{\phi(Y\,|X,\hat{a})}{\phi(Y\,|X,a)}\,\Big|\,\frac{f(X,\hat{a})}{f(X,a)}] = 1.$$

By the concavity of log(.), the only possibility is that $\frac{\phi(y|x,\hat{a})}{\phi(y|x,a)} = 1$, for almost every (x, y) and any $(\hat{a}, a)$, which means that X is a sufficient statistic of (X, Y). ■

This result may not be surprising. If X is a sufficient statistic of (X, Y), then Y does not contain any additional information. The additional term $\frac{f(x,y,\hat{a})}{f(x,y,a)}$ does not make difference because both the local and global likelihood ratios do not vary over Y. Therefore, any incentive compatible mechanism based on (X, Y), there is an equivalent mechanism based on X implementing the same effort as well.[14] So we extend Holmstrom's Proposition 3 (under risk-neutrality) or Kim's Proposition 2 .

It is also interesting to check whether Blackwell's condition is still sufficient. Blackwell's theorem states that any decision maker prefers signal X to Y *if and only if* there exists a nonnegative function $\phi(x, y)$ such that

$$\begin{aligned} f(y,a) &= \int \phi(x,y) f(x,a)dx \text{ for any } y \text{ and } a, \qquad \text{(BC)} \\ \int \phi(x,y)dy &= 1 \text{ for any } x, \\ 0 &< \int \phi(x,y)dx < \infty \text{ for any } y. \end{aligned}$$

To understand the issue, let us consider the case of single task with unidimensional signal first. With the FOA, Kim (1995) shows that Blackwell's condition implies that $l_a(X, a)$ is a MPS of $l_a(Y, a)$, then X is more efficient. Without the FOA, even if Blackwell's condition implies that $\frac{f(X,\hat{a})}{f(X,a)}$ is a MPS of $\frac{f(Y,\hat{a})}{f(Y,a)}$ as well, it is not guaranted to have that

$$(l_a(X,a), \frac{f(X,\hat{a})}{f(X,a)}) \text{ is a MPS of } (l_a(Y,a), \frac{f(Y,\hat{a})}{f(Y,a)}).$$

The extra conditions we may need are

$$\mathbb{E}[\frac{f(X,\hat{a})}{f(X,a)}\,|l_a(Y,a)] = \frac{f(Y,\hat{a})}{f(Y,a)}$$

or

$$\mathbb{E}[\ l_a(X,a)|\,\frac{f(Y,\hat{a})}{f(Y,a)}] = l_a(Y,a).$$

These conditions mean that the global likelihood ratio $\frac{f(Y,\hat{a})}{f(Y,a)}$ should not contain any more information than the local likelihood ratio $l_a(Y, a)$, and vice versa. We state the following conclusion.

[14] Recall that in Holmstrom (1979) Proposition 3, the sufficient part essentially does not depend on the validity of the FOA, while the necessary part does.

Proposition 6 *Assume that conditions in Proposition 4 hold. For $\mathbb{A} \subset \mathbb{R}$ and $Y \in \mathbb{R}$, if $f(y,a)$ satisfies MLRP, then Blackwell's conditon is sufficient for the information system X to be more efficient than Y.*

Proof. Using the same technique as that of Kim (1995, Proposition 4), when Blackwell's condition holds, $l_a(X,a)$ is a MPS of $l_a(Y,a)$. The global likelihood ratio $\frac{f(X,\hat{a})}{f(X,a)}$ is also a MPS of $\frac{f(Y,\hat{a})}{f(Y,a)}$. Therefore, by MLRP, we have

$$\mathbb{E}[\frac{f(X,\hat{a})}{f(X,a)}\,|l_a(Y,a)] = \mathbb{E}[\frac{f(X,\hat{a})}{f(X,a)}\,|Y] = \mathbb{E}[\frac{f(X,\hat{a})}{f(X,a)}\,\middle|\frac{f(Y,\hat{a})}{f(Y,a)}] = \frac{f(Y,\hat{a})}{f(Y,a)},$$

which yields

$$\mathbb{E}[(l_a(X,a),\frac{f(X,\hat{a})}{f(X,a)})\,|l_a(Y,a)] = (l_a(Y,a),\frac{f(Y,\hat{a})}{f(Y,a)}).$$

The desired conclusion is obtained. ■

Proposition 6 implies that Blackwell's condition alone in general may not be sufficient for the efficiency ranking (it is not necessary either by Kim (1995)). Its sufficiency based on the FOA seems not robust, because we need some global conditions even if the principal wants to implement a specific action. MLRP helps in the sense that all information is contained by the local likelihood ratio. Proposition 6 reminds us that without taking full care of the agent's IC constraint, we may understate the sufficient condition for X to be more efficient.

When we move to multi-task and multi-signal scenario, the question becomes more complicated. Even is MLRP not sufficient because the local likelihood ratio does not contain all information, when there are interactions between signals and tasks. Blackwell's condition and MLRP does not imply $\mathbb{E}\,[l_a(X,a)\,|l_a(Y,a)] = l_a(Y,a)$, a multivariate MPS condition. Moreover, the global likelihood ratio $\frac{f(.,\hat{a})}{f(.,a)}$ also bring new complications. The following lemma states a sufficient conditions.

Lemma 6 *If X and Y satisfy condition*

$$\mathbb{E}\left[l_a(X,a)\,\middle|l_a(Y,a')\right] = l_a(Y,a), \text{ for any } a,\ a' \in \mathbb{A}, \tag{16}$$

then,

$$\mathbb{E}\left[(l_a(X,a),\frac{f(X,\hat{a})}{f(X,a)})\,|l_a(Y,a)\right] \geq (l_a(Y,a),\frac{f(Y,\hat{a})}{f(Y,a)}).$$

In particular, when Blackwell's condition holds, $l_a(X,a)$ and $l_a(Y,a)$ are random vectors with independent components, and the matrix $l_a(y,a)$ is differentiable in y and $l_{ay'}(y,a)$ is of full rank, for almost every y and any $a \in \mathbb{A}$, then condition (16) is satisfied.

Proof. We only need to show

$$\mathbb{E}[\frac{f(X,\hat{a})}{f(X,a)}\,|l_a(Y,a)] = \mathbb{E}\,(\exp(\log f(X,\hat{a}) - \log f(X,a))\,|l_a(Y,a)\,) \geq \frac{f(Y,\hat{a})}{f(Y,a)}.$$

Taking piecewise difference, we have

$$
\begin{aligned}
& \log f(X,\hat{a}) - \log f(X,a) \\
= \;& \log f(X,\hat{a}_1,\hat{a}_2,...,\hat{a}_{N-1},\hat{a}_N) - \log f(X,a_1,\hat{a}_2,...,\hat{a}_{N-1},\hat{a}_N) \\
& + \log f(X,a_1,\hat{a}_2,...,\hat{a}_{N-1},\hat{a}_N) - \log f(X,a_1,a_2,...,\hat{a}_{N-1},\hat{a}_N) \\
& +... + \log f(X,a_1,a_2,...a_{N-1},\hat{a}_N) - \log f(X,a_1,a_2,...a_{N-1},a_N) \\
= \;& \int_{a_1}^{\hat{a}_1} l_{a_1}(X,\tau_1,\hat{a}_2,...,\hat{a}_N)d\tau_1 + \int_{a_2}^{\hat{a}_2} l_{a_2}(X,a_1,\tau_2,\hat{a}_3...,\hat{a}_N)d\tau_2 + ... + \int_{a_N}^{\hat{a}_N} l_{a_2}(X,a_1,a_2,...,a_{N-1},\tau_N)d\tau_N.
\end{aligned}
$$

By condition (16),

$$\mathbb{E}[l_{a_1}(X,\tau_1,\hat{a}_2,...,\hat{a}_N)\,|l_{a_1}(Y,a)] = l_{a_1}(Y,\tau_1,\hat{a}_2,...,\hat{a}_N)$$

And similar reasonings apply to $l_{a_2}(.)$, $l_{a_3}(.)$, and so on. We have

$$
\begin{aligned}
& \mathbb{E}[(\log f(X,\hat{a}) - \log f(X,a))\,|l_a(Y,a)] \\
= \;& \int_{a_1}^{\hat{a}_1} l_{a_1}(Y,\tau_1,\hat{a}_2,...,\hat{a}_N)d\tau_1 + \int_{a_2}^{\hat{a}_2} l_{a_2}(Y,a_1,\tau_2,\hat{a}_3...,\hat{a}_N)d\tau_2 + ... + \int_{a_N}^{\hat{a}_N} l_{a_2}(Y,a_1,a_2,...,a_{N-1},\tau_N)d\tau_N \\
= \;& \log f(Y,\hat{a}) - \log f(Y,a),
\end{aligned}
$$

yielding

$$\mathbb{E}[\exp(\log f(X,\hat{a}) - \log f(X,a))\,|l_a(Y,a)] \geq \exp\{\mathbb{E}[(\log f(X,\hat{a}) - \log f(X,a))\,|l_a(Y,a)]\} = \frac{f(Y,\hat{a})}{f(Y,a)}.$$

Now we show the second part. When Blackwell's condition holds, we have that

$$\mathbb{E}\min\{l_{a_i}(Y,a) - t_i, 0\} \geq \mathbb{E}\min\{l_{a_i}(X,a) - t_i, 0\}$$

for any t_i in $l_{a_i}(Y,a)$'s support. Therefore, $\mathbb{E}\left[l_{a_i}(X,a)\,|l_{a_i}(Y,a)\right] = l_{a_i}(Y,a)$ for $i = 1,.2..,N$. Because $l_a(X,a)$ and $l_a(Y,a)$ are random vectors with independent components, therefore, we have

$$\mathbb{E}\left[l_a(X,a)\,|l_a(Y,a)\right] = l_a(Y,a).$$

note that $l_{ay'}(y,a)$ is of full rank for almost every y and any $a \in \mathbb{A}$, therefore, the equation

$$l_a(Y,a) = l_a(y,a)$$

has a unique solution $Y = y$. We have

$$l_a(Y,a) = \mathbb{E}\left[l_a(X,a)\,|l_a(Y,a)\right] = \mathbb{E}\left[l_a(X,a)\,|Y\right] = \mathbb{E}\left[l_a(X,a)\,\middle|l_a(Y,a')\right].$$

■

A straightforward corollary is stated as follows.

Corollary 3 *Assume that conditions in Proposition 4 hold. For $\mathbb{A} \subset \mathbb{R}$ and $Y \in \mathbb{R}$, if $f(y,a)$ satisfies MLRP, then X is more efficient than Y if and only if $l_a(X,a)$ is a MPS of $l_a(Y,a)$.*

Proof. The sufficiency is proved by Proposition 6. The necessity is obvious. ■

Compared with Kim's (1995) main results, Corollary 3 replaces the controversial clause such as "assuming that the FOA is valid" by a clear-cutting condition, i.e., MLRP. Note that limiting the comparison to the distributions under which the FOA is valid or the agent's utility is globally concave seems restrictive. For example, one needs MLRP together with CDFC to assure the global concavity of the agent's utility (Rogerson, 1985). Corollary 3 at least removes CDFC constraint.[15]

Condition (16) can be regarded as multidimensional generalization of MLRP and the MPS condition. This condition is more demanding than the MPS condition because it requires the conditional mean to be invariant over $l_a(y,a')$ for all $a' \in \mathbb{A}$. Under this global property, we obtain the following conclusion.

Proposition 7 *Assume that conditions in Proposition 4 hold. Then, if w or $u(.)$ is bounded from the above, $f(y,a)$ satisfies condition (16), then information system Y is more efficient than X.*

Proof. When w or $u(.)$ is uniformly bounded from the above, say $\overline{w}$ or $u(\overline{w})$, the reasoning of Proposition 4 does not change by changing the utility function to $\tilde{u}(w) = u(w) - u(\overline{w}) \leq 0$. Therefore, $\min_w[w - q\tilde{u}(w)]$ is a increasing and concave function of q. For any increasing and concave function $\Phi(.)$, we have

$$\mathbb{E}\Phi(q(X)) = \mathbb{E}[\mathbb{E}_{X|l_a(Y,a)}[\Phi(q(X))\,|l_a(Y,a)]] \leq \mathbb{E}\Phi(\mathbb{E}_{X|l_a(Y,a)}[q(X)\,|l_a(Y,a)]) \leq \mathbb{E}\Phi(q(Y)).$$

where the last step is by Lemma 6,

$$\mathbb{E}_{X|l_a(Y,a)}[q(X)\,|l_a(Y,a)] = \lambda + \mu \cdot l_a(Y,a) + \delta(1 - \mathbb{E}[\frac{f(Y,\hat{a})}{f(Y,a)}\,|l_a(Y,a)]) \leq q(Y).$$

■

The following useful exponential family distribution satisfies condition (16):

$$f(y,a) = \frac{\exp(\sum_{i=1}^N a_i\eta_i(y))\kappa(y)}{\int \exp(\sum_{i=1}^N a_i\eta_i(y))\kappa(y)dy}, \tag{17}$$

where $\eta_i(y)$ is a real function for $i = 1,2,..,N$, $\kappa(y) > 0$ and $\int \exp\{\sum_{i=1}^N a_i y_i\}\kappa(y)dy < \infty$. For this family, we have

$$l_{a_i}(Y,a) = \eta_i(Y) - \mathbb{E}\eta_i(Y).$$

[15] The CDFC is only a sufficient condition for the validity of the FOA. But so far we do not have more general sufficient condition than the CDFC, without the additional restriction on $u(.)$.

Then, $\mathbb{E}\left[l_a(X,a)\,|l_a(Y,a)\right] = \mathbb{E}\left[l_a(X,a)\,|\eta(Y)\right] = \mathbb{E}\left[l_a(X,a)\,|l_a(Y,a')\right]$, which implies condition (16).

We go back to Blackwell's condition. Even for family (17), Blackwell's condition is not sufficient. Additional conditions are $\eta_i(y) = y_i$ and that $l_a(X,a)$ and Y have independent components, which are more restrictive than condition (16).

A more general condition without an upper bound of the utility is based on the lift zonoid representation of the probability measure. Define (Mosler, 2002, Definition 2.2) the lift zonoid of an integrable likelihood ratio $(l_a(x,a), \frac{f(x,\hat{a})}{f(x,a)}) \in \mathbb{R}^{N+1}$ as

$$\mathcal{Z}((l_a(X,a), \frac{f(X,\hat{a})}{f(X,a)})) \equiv \left\{ \left(\int g(x)d\Psi(x), \int xg(x)d\Psi(x) \right) |g : \mathbb{R}^{N+2} \to [0,1] \text{ measurable} \right\},$$

where $\Psi(x)$ is the joint cummulative distribution function of random vector $(l_a(X,a), \frac{f(X,\hat{a})}{f(X,a)})$.

According to Mosler (2002, Definition 8.1), the *lift zonoid order* means the two random vectors satisfy the following property:[16]

$$\mathcal{Z}(l_a(Y,a), \frac{f(Y,\hat{a})}{f(Y,a)}) \subset \mathcal{Z}((l_a(X,a), \frac{f(X,\hat{a})}{f(X,a)})). \tag{18}$$

Assuming the global concavity of the $U(w_{\lambda,\mu}(.,a),\tilde{a})$ in $\tilde{a}$ or the validity of the FOA, Xie (2011) shows that signal X is more efficient at a than signal Y if and only if

$$\mathcal{Z}(l_a(Y,a)) \subset \mathcal{Z}((l_a(X,a))), \tag{19}$$

which is called the lift zonoid criterion. The lift zonoid criterion is more general than the MPS criterion, unless the task is unidimensional or the likelihood ratio is normally distributed (see Xie, 2011).

Without the FOA, the following proposition states a necessary and sufficient condition for information ranking using the lift zonoid order.

Proposition 8 *Assume that conditions in Proposition 4 hold. For two signals X and Y, X is more efficient than Y if and only if (18) holds.*

Proof. Note that $q(x)$ is a linear function of $(l_a(x,a), \frac{f(x,\hat{a})}{f(x,a)})$, then according to Theorem 8.5 in Mosler (2002), (G-MPS) holds if and only if (18) holds. ■

4.2 Curvature of the Optimal Contract

The analysis of the curvature of optimal contract is also plausible. One basic concern is the monotonicity of the payment schedule. Under MLRP, if the FOA is valid, the payment is monotone

[16] According to Mosler (2002), the lift zonoid order is more general than the increasing convex order (the expecation of any increasing convex function of random vector X is less than that of Y)

in output (see, e.g., Milgrom, 1982; Hart and Holmstrom, 1987). If the FOA is invalid, whether or not the schedule is monotone may depend on the relationship between the optimal action a^{**} and its alternative best response $\hat{a}^*$. We take the one-task case with a risk-neutral principal as an example.

Proposition 9 *Assume that $u(.)$ is concave and $v(.)$ is weakly concave. For unidimensional $a \in \mathbb{A} \subset \mathbb{R}$, assume also $c'(a) > 0$ and that for $a' > a$, as $l_a(x,a) \to \infty$, $\frac{l_a(x,a')}{l_a(x,a)^\alpha}$ converges to a positive constant for some $\alpha \in (0,\infty)$. Then, under $w^*(x, a^{**}, \hat{a}^*)$, (i) if $\hat{a}^* > a^{**}$, then when $l_a(x,a^{**}) \to \infty$, $\mu^* > 0$ and $w^*(x, a^{**}, \hat{a}^*)$ is decreasing in x. (ii) If $\hat{a}^* < a^{**}$, when $l_a(x,a^{**}) \to \infty$, μ^* may be either positive or negative, and $w^*(x, a^{**}, \hat{a}^*)$ may be either decreasing or increasing.*

Proof. We first consider the case in which $\hat{a}^* > a^{**}$. By MLRP, $\frac{f(x,\hat{a}^*)}{f(x,a^{**})}$ is nondecreasing. We show $\mu^* > 0$ by contradiction. Suppose $\mu^* < 0$, then $w^*(x, a^{**}, \hat{a}^*)$ is decreasing, therefore

$$\int u(w^*(x, a^{**}, \hat{a}^*)) f_a(x, a^{**}) dx < 0,$$

which implies that $a^{**} = \underline{a}$ is the unique best response, a contradiction. Given $\mu^* > 0$, note that $\frac{f(x,\hat{a}^*)}{f(x,a^{**})} = e^{(\hat{a}^* - a^{**}) l_a(x,a')}$ for some $a^{**} \le a' \le \hat{a}^*$. Then, as $l_a(x,a^{**}) \to \infty$, the term $e^{(\hat{a}^* - a^{**}) l_a(x,a')}$ dominates $l_a(x,a^{**})$, since $\frac{l_a(x,a')}{l_a(x,a^*)^\alpha}$ converges to a positive constant. Therefore,

$$\lambda^* + \mu^* l_a(x, a^{**}) + \delta^* \left(1 - e^{(\hat{a}^* - a^{**}) l_a(x,a')}\right)$$

is decreasing for a sufficiently large large x.

Next, in the case of $\hat{a}^* < a^{**}$, as the sign of μ^* is indeterminant, we only check the curvature when $l_a(x,a^{**}) \to \infty$. In this case, $l_a(x,a^{**})$ dominates $e^{(\hat{a}^* - a^{**}) l_a(x,a')}$, and $w^*(x, a^{**}, \hat{a}^*)$ can be either decreasing or increasing. ■

The intuition is that, when the agent has an undesirable alternative best response that is higher than the targeted optimal effort, the principal will likely set a non-monotone incentive scheme even if MLRP holds. The source of non-monotonicity comes from the iterm $(1 - \frac{f(x,\hat{a})}{f(x,a)})$ when $\hat{a} > a$. The prediction here is consistent with Grossman and Hart's (1983) Proposition 5. However, the analysis that is based on the visible functional form provides a more transparent interpretation.

In particular, if $f(x,a)$ belongs to exponential family (17), then the optimal contract has a nice form

$$w_{\lambda,\mu,\delta}(x, a, \hat{a}) = r\left(\lambda + \sum_{i=1}^{N} (\eta_i(x) - \mathbb{E}\eta_i(X)) + \delta\left(C_1 \exp(\sum_{i=1}^{N} (a_i - \hat{a}_i)\eta_i(x)) - 1\right), \pi\right),$$

where $r(q,\pi)$ solves equation $\frac{v'(\pi - w)}{u'(w)} = q$ for w and $C_1 > 0$ is a constant. Clearly, the first argument in $r(.,.)$ is convex in $(\eta_1(x), ..., \eta_N(x))$. The slope of $w_{\lambda,\mu,\delta}(x, a, \hat{a})$ in terms of $\eta_i(x)$ may also depend on the sign of $(a_i - \hat{a}_i)$.

5 Moral General Utility Structure

Consider a more general case in which the principal's Bernoulli utility is $v(w, x, a)$ and the agent's Bernoulli utility is $u(w, x, a)$. We assume that $v_w(., x, a) < 0$ and $u_w(., x, a) > 0$, so there is a conflict of interest between the two parties. Then, the principal's expected utility is

$$V(w, a) = \int v(w(x), x, a) f(x, a) dx$$

and the agent's expected utility is

$$U(w, a) = \int u(w(x), x, a) f(x, a) dx,$$

where we only consider a deterministic contract $w(x)$.

The Lagrangian (1) becomes

$$\mathcal{L}(w, a, \hat{a}; \lambda, \mu, \delta; U) = \int L(w, a, \hat{a}; \lambda, \mu, \delta; U) f(x, a) dx,$$

where

$$\begin{aligned} & L(w, a, \hat{a}; \lambda, \mu, \delta; U) \\ = \; & v(w, x, a) + \lambda(u(w, a) - U) + \mu \cdot [u(w, x, a) l_a(x, a) + u_a(w, x, a)] + \delta(u(w, x, a) - \frac{f(x, \hat{a})}{f(x, a)} u(w, \hat{a})). \end{aligned}$$

The AMH contract class is characterized by the first-order condition with respect to w (for almost every x)

$$-\frac{v_w(w, x, a)}{u_w(w, x, a)} = \lambda + \mu \cdot [l_a(x, a) + \frac{u_{wa}(w, x, a)}{u_w(w, x, a)}] + \delta(1 - \frac{f(x, \hat{a})}{f(x, a)} \frac{u_w(w, x, \hat{a})}{u_w(w, x, a)}), \tag{20}$$

whenever $w \geq \underline{w}$; otherwise $w = \underline{w}$. The solution to (20) usally is no longer unique, even if $u(., x, a)$ or $v(., x, a)$ is concave. We denote the solution set as $W_{\lambda,\mu,\delta}(x, a, \hat{a})$ and for a bit abuse of notation, we still use the notation $w_{\lambda,\mu,\delta}(x, a, \hat{a}) \in W_{\lambda,\mu,\delta}(x, a, \hat{a})$ to denote a selection of AMH contract.[17]

Despite of the non-uniqueness of the AMH contract, the existence of a suitable Lagrangian multiplier (λ, μ, δ) is guranteed by the conflicting preference assumption. We first extend Lemmas 1 and 3.

Proposition 10 *Assume $v_w < 0$, $u_w > 0$, and that the principal-agent problem has a deterministic solution. Then, for any implementable $\tilde{a}^*$, the optimal contract implementing $\tilde{a}^*$ and delivering at least utility U to the agent is a selection of AMH contract*

$$w^*(x, \tilde{a}^*, \hat{a}^*; U) \equiv w_{\lambda^*,\mu^*,\delta^*}(x, \tilde{a}^*, \hat{a}^*) \in W_{\lambda(\tilde{a}^*,\hat{a}^*;U),\mu(\tilde{a}^*,\hat{a}^*;U),\delta(\tilde{a}^*,\hat{a}^*;U)}(x, \tilde{a}^*, \hat{a}^*)$$

[17] $w_{\lambda,\mu,\delta}(x, a, \hat{a})$ should be bounded almost everywhere, but there may be multiple optimal solutions.

where (i) $(\lambda^*, \mu^*, \delta^*) \in (\lambda(\tilde{a}^*, \hat{a}^*; U), \mu(\tilde{a}^*, \hat{a}^*; U), \delta(\tilde{a}^*, \hat{a}^*; U))$ *is a selection of the Lagrangian multiplier at* $(a, \hat{a}) = (\tilde{a}^*, \hat{a}^*)$*, in which* $(\lambda(a, \hat{a}; U), \mu(a, \hat{a}; U), \delta(a, \hat{a}; U))$ *denotes a set of* (λ, μ, δ) *solving the complementary slackness conditions*

$$
\begin{aligned}
(a)\ \lambda &\geq 0,\ U(w_{\lambda,\mu,\delta}(., a, \hat{a}), a) - U \geq 0 \text{ and } \lambda[U(w_{\lambda,\mu,\delta}(., a, \hat{a}), a) - U] = 0, \\
(b)\ 0 &= U_a(w_{\lambda,\mu,\delta}(., a, \hat{a}), a), \quad \text{and} \\
(c)\ \delta &\geq 0,\ U(w_{\lambda,\mu,\delta}(., a, \hat{a}), a) - U(w_{\lambda,\mu,\delta}(., a, \hat{a}), \hat{a}) \geq 0 \text{ and } \delta[U(w_{\lambda,\mu,\delta}(., a, \hat{a}), a) - U(w_{\lambda,\mu,\delta}(., a, \hat{a}), \hat{a})] = 0;
\end{aligned}
$$

and (ii)

$$\hat{a}^* \in \arg\min_{\hat{a}} V(w^*(., \tilde{a}^*, \hat{a}; U), \tilde{a}^*).$$

In particular, $w^*(., \tilde{a}^*, \hat{a}^*; U)$ *is a stationary point of* $L(w, a^*, \hat{a}^*; \lambda^*, \mu^*, \delta^*; U)$ *over* w*.*

Proof. Suppose there is an optimal contract $\tilde{w}^*$ implementing $\tilde{a}^*$ and delivering at least utility U to the agent. For every deviation $h(x) \geq 0$, with $h(\underline{x}) = h(\overline{x}) = 0$, we follow the same construction as in the Proof of Lemma 3. Consider a contract

$$\tilde{w} = \tilde{w}^* + zh,$$

for $z \in \mathbb{R}$. We want to show that $z = 0$ is the optimal choice.

Use the same short notations $V(z, a) = V(\tilde{w}^* + zh, a)$, $U(z, a) = U(\tilde{w}^* + zh, a)$ and $\xi^{RIC}(a) = \{a : (\tilde{w}, a) \text{ satisfy } (RIC)\}$. By the conflicting preference $v_w < 0$ and $u_w > 0$, we obtain

$$\inf_{\hat{a}} \max_{z} \{V(z, \tilde{a}^*) : U(z, \tilde{a}^*) \geq U,\ z \in \xi^{RIC}(\tilde{a}^*),\ \text{and } U(z, \tilde{a}^*) - U(z, \hat{a}) \geq 0\} = V(\tilde{w}^*, \tilde{a}^*).$$

Let

$$\hat{a}^* \in \arg\inf_{\hat{a}} \max_{z} \{V(z, \tilde{a}^*) : U(z, \tilde{a}^*) \geq U,\ z \in \xi^{RIC}(\tilde{a}^*),\ \text{and } U(z, \tilde{a}^*) - U(z, \hat{a}) \geq 0\}$$

be a solution of the above problem, we also have

$$0 \in \arg\max_{z} \{V(z, \tilde{a}^*) : U(z, \tilde{a}^*) \geq U, z \in \xi^{RIC}(\tilde{a}^*),\ \text{and } U(z, \tilde{a}^*) - U(z, \hat{a}^*) \geq 0\}.$$

Therefore, the maximizer $z^* = 0$ should satisfy the necessary conditions

$$V_z(z^*, \tilde{a}^*) + \lambda[U_z(z^*, \tilde{a}^*) - U] + \mu \cdot U_{az}(z^*, a^*) + \delta[U_z(z^*, \tilde{a}^*) - U_z(z^*, \hat{a}^*)] = 0,$$

and the three constraints are also satisfied for some (λ, μ, δ), where $U_z(z, \tilde{a}^*)$, $U_{az}(z, a)$, and $U_z(z, \tilde{a}^*) - U_z(z, \hat{a}^*)$ denote the partial derivatives. Thus, $\tilde{w}^*$ is a stationary point of $\mathcal{L}(w, \tilde{a}^*, \hat{a}^*, \lambda, \mu, \delta; U)$ over w and we obtain the first-order condition (20) at $(a, \hat{a}) = (\tilde{a}^*, \hat{a}^*)$, which characterizes $\tilde{w}^*$ as a selection of $W_{\lambda,\mu,\delta}(x, a, \hat{a})$. Therefore, $\tilde{w}^*$ is an AMH contract $w_{\lambda,\mu,\delta}(x, \tilde{a}^*, \hat{a}^*)$, where

(λ, μ, δ) should be determined by the three constraints $U(\tilde{w}^*, \tilde{a}^*) \geq U^*$, $U_a(\tilde{w}^*, \tilde{a}^*) = 0$, and $U(\tilde{w}^*, \tilde{a}^*) - U(\tilde{w}^*, \hat{a}^*) \geq 0$ with the complementary slackness conditions. Since the Lagrangian multiplier may not be unique, let $(\lambda^*(\tilde{a}^*, \hat{a}^*; U), \mu^*(\tilde{a}^*, \hat{a}^*; U), \delta^*(\tilde{a}^*, \hat{a}^*; U))$ denote the multiplier set. There exists a selection $(\lambda^*, \mu^*, \delta^*) \in (\lambda^*(\tilde{a}^*, \hat{a}^*; U), \mu^*(\tilde{a}^*, \hat{a}^*; U), \delta^*(\tilde{a}^*, \hat{a}^*; U))$ such that

$$\tilde{w}^* = w_{\lambda^*, \mu^*, \delta^*}(x, \tilde{a}^*, \hat{a}^*) \equiv w^*(., \tilde{a}^*, \hat{a}^*; U).$$

Accordingly, using the same argument as in Lemma 2 again, $\hat{a}^*$ is chosen by the minimization problem

$$\min_{\hat{a}} V(w^*(., \tilde{a}^*, \hat{a}; U), \tilde{a}^*).$$

After solving (λ, μ, δ) and $\hat{a}$ (in terms of $\tilde{a}^*$ and U), $w^*(., \tilde{a}^*, \hat{a}^*; U)$ is a stationary point of $L(w, a^*, \hat{a}^*; \lambda^*, \mu^*, \delta^*; U)$ over w. ■

There are several notable differences between Lemma 3 and Proposition 10. First, $w^*(x, \tilde{a}^*, \hat{a}^*; U)$ may not be a global maximizer of $\mathcal{L}(w, a^*, \hat{a}^*; \lambda, \mu, \delta; U)$ over w and the Lagrangian multiplier vector may not be unique. Second, $w^*(x, \tilde{a}^*, \hat{a}^*; U)$ may not even be a local maximizer of

$$\max_{w} \{V(w, \tilde{a}^*) : U(w, \tilde{a}^*) \geq U, \ U_a(w, \tilde{a}^*) = 0, \text{ and } U(w, \tilde{a}^*) - U(w, \hat{a}) \geq 0\},$$

because in our proof, the deviation $h(x)$ is restricted to be nonnegative, which is weaker than every possible deviation. What we can say is that $w^*(x, \tilde{a}^*, \hat{a}^*; U)$ cannot be a local minimizer.[18]

As a result, the global maximizer of problem $(\mathrm{P}|a, \hat{a}; U)$, say, w^{GM}, may cross $\tilde{w}^*$ in several places, although $w^{GM} - \tilde{w}^*$ cannot be positive or negative a.e. If further conditions are added, it may be plausible to show that $\tilde{w}^*$ is a local maximum.

Despite these weaknesses, Proposition 10 at least provides a characterization for the optimal contract under a very general environment. To close the problem, the next step is to find the optimal implementable $\tilde{a}^*$ and the utility level U^*, similar to Section 4. We can search for the optimal action that is implementable by a general AMH contract $w_{\lambda,\mu,\delta}(x, a, \hat{a})$ and delivers the agent utility U^*.

Proposition 11 *Assume that $v_w < 0$, $u_w > 0$, and that the principal-agent problem has a deterministic solution (w^{**}, a^{**}), then*

$$V(w^{**}, a^{**}) = \max_{a} \min_{\hat{a}} V(w^*(., a, \hat{a}, U^*), a),$$

[18] If $\tilde{w}^*$ is a local minimum, then there exists some deviation $h(x) \geq 0$ and $z' \neq 0$ such that $\tilde{w}^* + zh$ gives the principal higher utility and all constraints are satisfied. This is a contradiction to that $z^* = 0$ is the maximizer of $V(z, \tilde{a}^*)$ within the constrained set.

where $w^*(.,a,\hat{a},U^*)$ is a stationary point of $L(w,a,\hat{a};\lambda,\mu,\delta;U^*)$, which is specified by (20) and satisfies (IR), (RIC), and (NJ($\hat{a}$)) constraints. In addition,*

$$U^* = \min\left\{\arg\min_{U\geq \underline{U}}\left(V(\tilde{w}^*(U),\tilde{a}^*(U)) - \max_{a\in a^{BR}(\tilde{w}^*(U))} V(\tilde{w}^*(U),a)\right)\right\}, \tag{21}$$

where $\tilde{w}^*(U)$ and $V(\tilde{w}^*(U),\tilde{a}^*(U))$ are defined by

$$V(\tilde{w}^*(U),\tilde{a}^*(U)) = \max_a \min_{\hat{a}} V(w^*(.,a,\hat{a},U),a).$$

Proof. By Proposition 10, any implementable contract $\tilde{a}^* \in a^{BR}(\tilde{w}^*)$ such that $U = U(\tilde{w}^*,\tilde{a}^*)$ can be implemented by a general contract $w^*(.,\tilde{a}^*,\hat{a}^*;U)$ that solves (20). Therefore, given the agent's equilibrium utility level $U^* = U(w^{**},a^{**})$, by the same argument as in Lemma 4, the optimal incentive compatible action a^{**} can be found by

$$V(w^*(.,a^{**},\hat{a}^*,U^*),a^{**}) = \max_a \min_{\hat{a}} V(w^*(.,a,\hat{a},U^*),a)$$

and $w^{**} = w^*(.,a^{**},\hat{a}^*;U^*)$ is the optimal contract implementing a^{**}, where $w^*(.,a^{**},\hat{a}^*;U^*)$ satisfies (IR*), (RIC), and (NJ($\hat{a}^*$)). Finally, the agent's utility at optimum, say, U^* can be chosen to close the gap

$$V(\tilde{w}^*(U),\tilde{a}^*(U)) - \max_{a\in a^{BR}(\tilde{w}^*(U))} V(\tilde{w}^*(U),a),$$

using the same argument as in Lemma 5. ∎

Remark 4 *An alternative way to solve the problem given U^* is*

$$\max_{\lambda,\mu,\delta}\max_a \inf_{\hat{a}}\{V(w_{\lambda,\mu,\delta}(.,a,\hat{a}),a) : (w_{\lambda,\mu,\delta}(.,a,\hat{a}),a) \text{ satisfies (IR*), (RIC), and (NJ)}\},$$

where we search for the optimal parameter $(\lambda^*,\mu^*,\delta^*)$.

6 Conclusion

This paper develops a general method for solving principal-agent problems, regardless of the validity of the FOA. We show that there exists an augmented MH contract with only one additional no-jumping constraint that is optimal. This is the most parsimonious contract form that we can expect, despite the topological structure of the best-response mapping. Therefore, we establish a general characterization of an optimal contract without the FOA, which allows us to do a comparative static analysis based on a simpler functional form of the contract. We provide a foundation for comparing the efficiency of information system without the FOA. The use of the approach developed here is not limited to moral hazard problems. We speculate that our method can shed light on solving a more general optimization problem with an IC constraint, such as dynamic principal-agent problems (not limited to moral hazard), and even lead to an algorithm for general bi-level optimization problems.

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7 Appendix

7.1 A1. Proof of Lemma 1

Proof. For every $(a, \hat{a})$ and (λ, μ, δ), the AMH contract $w_{\lambda,\mu,\delta}(x, a, \hat{a})$ specified by formula (2) is differentiable in each arguments. Note that the Lagrangian dual

$$\Phi(\lambda, \mu, \delta \,|a, \hat{a}\,) \equiv \mathcal{L}(w_{\lambda,\mu,\delta}(., a, \hat{a}), a, \hat{a}; \lambda, \mu, \delta; U)$$

is continuous and differentiable in $(a, \hat{a}; \lambda, \mu, \delta)$. Moreover, given $(a, \hat{a})$, $\Phi(\lambda, \mu, \delta \,|a, \hat{a}\,)$ is known to be globally convex in (λ, μ, δ). Therefore, to show the existence and uniqueness of (λ, μ, δ), it suffices to show matrix

$$\Delta \equiv \begin{pmatrix} U_\lambda(w_{\lambda,\mu,\delta}, a) & U_\mu(w_{\lambda,\mu,\delta}, a) & U_\delta(w_{\lambda,\mu,\delta}, a) \\ U_{a\lambda}(w_{\lambda,\mu,\delta}, a) & U_{a\mu'}(w_{\lambda,\mu,\delta}, a) & U_{a\delta}(w_{\lambda,\mu,\delta}, a) \\ U_\lambda(w_{\lambda,\mu,\delta}, a) - U_\lambda(w_{\lambda,\mu,\delta}, \hat{a}) & U_\mu(w_{\lambda,\mu,\delta}, a) - U_\mu(w_{\lambda,\mu,\delta}, \hat{a}) & U_\delta(w_{\lambda,\mu,\delta}, a) - U_\delta(w_{\lambda,\mu,\delta}, \hat{a}) \end{pmatrix}$$

is of full rank, where $U_j(w_{\lambda,\mu,\delta}, a)$ denote the partial derivative w.r.t. $j = \lambda, \mu_1, ...\mu_N, \delta$.

Note that

$$\begin{aligned} U_\lambda(w_{\lambda,\mu,\delta}, a) &= \int u'(w_{\lambda,\mu,\delta}) \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi-w)}{u'(w)}\,|_{w=w_{\lambda,\mu,\delta}}} f(x,a)dx \\ U_{a\lambda}(w_{\lambda,\mu,\delta}, a) &= \int u'(w_{\lambda,\mu,\delta}) l_a \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi-w)}{u'(w)}\,|_{w=w_{\lambda,\mu,\delta}}} f(x,a)dx = U_\delta(w_{\lambda,\mu,\delta}, a) \\ U_\lambda(w_{\lambda,\mu,\delta}, a) - U_\lambda(w_{\lambda,\mu,\delta}, \hat{a}) &= \int u'(w_{\lambda,\mu,\delta})(1 - \frac{f(x,\hat{a})}{f(x,a)}) \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi-w)}{u'(w)}\,|_{w=w_{\lambda,\mu,\delta}}} f(x,a)dx = U_\delta(w_{\lambda,\mu,\delta}, a) \\ U_{a_i\mu_j} &= \int u'(w_{\lambda,\mu,\delta}) l_{a_i} l_{a_j} \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi-w)}{u'(w)}\,|_{w=w_{\lambda,\mu,\delta}}} f(x,a)dx \end{aligned}$$

and

$$U_{a\delta} = \int u'(w_{\lambda,\mu,\delta}) l_a (1 - \frac{f(x,\hat{a})}{f(x,a)}) \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi-w)}{u'(w)}\,|_{w=w_{\lambda,\mu,\delta}}} f(x,a)dx = U_\delta(w_{\lambda,\mu,\delta}, a) - U_\delta(w_{\lambda,\mu,\delta}, \hat{a}).$$

The matrix Δ can then be written as

$$\Delta = \mathbb{E}\theta\theta',$$

where $\theta' = (\theta_1, \theta_2, ..., \theta_{N+1}, \theta_{N+2})$ and

$$\begin{aligned} \theta_1 &= \sqrt{u'(w_{\lambda,\mu,\delta}) \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi-w)}{u'(w)}\,|_{w=w_{\lambda,\mu,\delta}}}}, \\ \theta_i &= \sqrt{u'(w_{\lambda,\mu,\delta}) \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi-w)}{u'(w)}\,|_{w=w_{\lambda,\mu,\delta}}}}\, l_{a_i} \quad (i = 2, 3, ..., N+1), \\ \theta_{N+2} &= \sqrt{u'(w_{\lambda,\mu,\delta}) \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi-w)}{u'(w)}\,|_{w=w_{\lambda,\mu,\delta}}}}\,(1 - \frac{f(x,\hat{a})}{f(x,a)}). \end{aligned}$$

Note that the covariance matrix $Cov(\theta, \theta')$ is positive definite unless θ_i and θ_j are perfectly linearly correlated. The pure linear correlation can be ruled out as follows. Suppose that they are perfectly linearly correlated, in the sense that $\theta_i = A_i + B_i\theta_1$ for each i, where A_i and B_i are constants. This perfect linear correlation also implies that at least one of the Lagrangian multipliers is unbounded. Then, we take a subsequence of $(\lambda^k, \mu^k, \delta^k)$ and divide both sides of the first-order condition (2) by $\lambda^k + \left\|\mu^k\right\| + \delta^k$. Thus, we have

$$\beta + \gamma \cdot l_a + \zeta(1 - \frac{f(x, \hat{a})}{f(x, a)}) = 0. \tag{22}$$

where $\beta = \lim_{k\to\infty} \frac{\lambda^k}{\lambda^k + \left\|\mu^k\right\| + \delta^k} \geq 0$, $\gamma = \lim_{k\to\infty} \frac{\mu^k}{\lambda^k + \left\|\mu^k\right\| + \delta^k}$ and $\zeta = \lim_{k\to\infty} \frac{\delta^k}{\lambda^k + \left\|\mu^k\right\| + \delta^k} \geq 0$, and (β, γ, ζ) cannot be all zeros.

From (22) and the linearity, we have

$$\beta\theta_1 + \sum_{i=2}^{N+1} \gamma_i(A_i + B_i\theta_1) + \zeta(A_{N+2} + B_{N+2}\theta_1) = 0,$$

which implies that θ_1 is constant a.e.. If $\theta_1 = \sqrt{u'(w_{\lambda,\mu,\delta}) \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi - w)}{u'(w)}\Big|_{w=w_{\lambda,\mu,\delta}}}}$ is a constant, it is impossible for $\sqrt{u'(w_{\lambda,\mu,\delta}) \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi - w)}{u'(w)}\Big|_{w=w_{\lambda,\mu,\delta}}}}\, l_{a_i}$ to be linearly correlated with $\sqrt{u'(w_{\lambda,\mu,\delta}) \frac{1}{\frac{\partial}{\partial w}\frac{v'(\pi - w)}{u'(w)}\Big|_{w=w_{\lambda,\mu,\delta}}}}$ because l_{a_i} can not be a constant. Therefore, the perfect linear correlation is impossible and $Cov(\theta, \theta')$ is positive definite.

Given that the covariance matrix $Cov(\theta, \theta')$ is positive definite, for any nonzero vector $Y \in \mathbb{R}^{N+2}$, we have

$$Y'Cov(\theta, \theta')Y = Y'(\mathbb{E}\theta\theta' - \mathbb{E}\theta\mathbb{E}\theta')Y = Y'[\mathbb{E}\theta\theta']Y - Y'(\mathbb{E}\theta\mathbb{E}\theta')Y > 0,$$

which implies $Y'[\mathbb{E}\theta\theta']Y > Y'(\mathbb{E}\theta\mathbb{E}\theta')Y \geq 0$. Thus, $\mathbb{E}\theta\theta'$ is positive definite, then is of full rank.

Furthermore, by the global convexity of $\Phi(\lambda, \mu, \delta\,|a, \hat{a})$ in (λ, μ, δ), the vector $(\lambda^*, \mu^*, \delta^*)$ that satisfies condition (ii) in Lemma 1 is the unique minimizer of $\Phi(\lambda, \mu, \delta\,|a, \hat{a})$. By the implicit function theorem, $(\lambda^*(a, \hat{a}; U), \mu^*(a, \hat{a}; U), \delta^*(a, \hat{a}; U))$ is continuous and differentiable (a.e.). ■

7.2 A2. Proof of Proposition 1

Proof. We only need to prove the sufficient condition for the IR constraint to be binding. The proof is essentially the same as in Grossman-Hart (1983, Proposition 11). Suppose to the contrary that (w^{**}, a^{**}) is an optimal contract in which (IR) is not binding. Consider the contract $\tilde{w}$ solving $u(\tilde{w}, a^{**}) = u(w^{**}, a^{**}) - \varepsilon$ for any constant $\varepsilon > 0$, whenever $w^{**} \neq \underline{w}$. We choose ε such that IR is binding. Note that if $a^{**} \in a^{BR}(w^{**})$,

$$U(\tilde{w}, a^{**}) = \int u(w^{**}, a^{**})dF(x, a^*) - \varepsilon \geq \int u(w^{**}, a)dF(x, a) - \varepsilon.$$

By the separability,

$$u(w^{**}, a) - u(\tilde{w}, a) \geq u(w^{**}, a^{**}) - u(\tilde{w}, a^{**}) = \varepsilon,$$

therefore, for any a,

$$U(\tilde{w}, a^{**}) = \int u(w^{**}, a^{**})dF(x, a^{**}) - \varepsilon \geq \int u(\tilde{w}, a)dF(x, a) + \varepsilon - \varepsilon = U(\tilde{w}, a),$$

implying that $\tilde{w}$ implements the same action as w^{**} does. As $\varepsilon > 0$, the principal is strictly better off with $\tilde{w}$, which is contradicts to the definition of w^{**}. ■

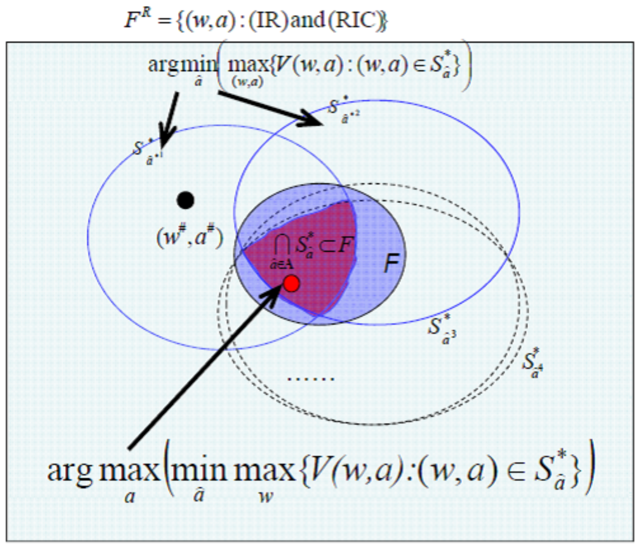

Figure 1. The Maxmin Representation of the Original Problem