A Model of the Yield Curve with Idiosyncratic Consumption Risk

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Abstract

This paper extends the standard consumption-based asset pricing model to a heterogeneous-agents framework. The key assumption is that agents are subject to both aggregate and uninsurable idiosyncratic risks. This leads to a pricing kernel that depends not only on aggregate per capita consumption growth and inflation, as in a conventional representative agent model, but also on the cross-sectional variance of individual consumption growth. The dynamics of the pricing kernel is modeled in a state-space representation that allows for maximal correlations among pricing factors. Under linearity and normality, the model falls within the broad class of essentially affine term structure models with a closed form solution of the yield curve. The maximum-likelihood estimation of the model using quarterly data on aggregate consumption growth, inflation and two nominal yields shows that the model can account for many salient features of the yield curve in the U.S.

Keywords: Yield Curve, Idiosyncratic Risk, Affine Term Structure Models

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1 Introduction

The US postwar data on zero coupon bond yields show some stylized facts: the average yield curve is upward sloping - the longer the maturity, the higher the yield; yields are highly autocorrelated, with increasing autocorrelations at longer maturities; yields of different maturities move together, especially for yields of near maturities.

This paper proposes a heterogeneous-agents consumption-based asset pricing model that accounts for all these key aspects. Two ingredients enable the model to capture these findings. The first is that agents are heterogeneous and heterogeneity lies in the fact that they are subject to uninsurable, persistent, idiosyncratic consumption risk.

The standard representative agent models assume complete market and full consumption insurance, therefore the corresponding empirical work focuses on aggregate per capita consumption and abstracts from idiosyncratic variations. However, realistically the complete market assumption is suspect since certain types of risks are largely uninsurable, such as, will I lose my job or not? will I be laid off or not?[1] Therefore, the real risk that agents face are a lot more than is reflected in the variations of aggregate consumption. This greater level of consumption risk makes agents more cautious about consuming today and increases their desire for precautionary saving. Thus agents who must bear both aggregate and idiosyncratic risks are willing to pay a higher price for transferring one unit of consumption from today to tomorrow, which makes bond prices increase and yields decrease.

Identifying idiosyncratic risk is key to matching the level of yields observed in the data, since yields predicted in a representative agent model with aggregate consumption alone are usually too high. For example, Piazzesi and Schneider (2006) have to use a large subjective discount factor ($\beta = 1.005$) to reduce yields predicted by a representative agent model with Epstein-Zin recursive preferences. Within the same representative agent framework but using power utility and robustness concern to model uncertainty, Xu (2008) finds that average nominal yields are on the order of 7.8 percent, while the data show,

[1] Tests by, for example, Cochrane (1991), Mace (1991) and Nelson (1994) also reject the full consumption insurance hypothesis.

for example, 5.15 percent for the yield on a 3-month nominal bond. With the presence of idiosyncratic risk, this paper successfully generates nominal yields that are close to the data level. The model also predicts reasonable yields for real bonds. For example, the 3-month real rate is around 1.4 percent.

In modeling idiosyncratic risk, I build on the framework of Constantinides and Duffie (1996). Their main result –no trade theorem – shows that if idiosyncratic shocks are persistent (follow a certain martingale process), agents will not find it useful to trade in assets to insure against such shocks. Therefore, idiosyncratic income shocks translate into idiosyncratic consumption risk, and the Euler equation of consumption in a representative agent framework is replaced by an Euler equation that depends not only on the aggregate per capita consumption growth but also on the cross-sectional variance of individual consumption growth. This gives rise to the relevant no-arbitrage pricing kernel in the economy. More specifically, it is composed of three factors: aggregate consumption growth, cross-sectional variance of individual consumption growth, and inflation when the empirical findings are usually in nominal terms.

Idiosyncratic consumption risk is clearly not enough. To capture the positive slope of the yield curve, we also need the second ingredient: a model describing agents' beliefs about the stochastic process for the pricing factors. I consider a specification where the dynamics of these three factors are modeled jointly in a state-space representation. Quarterly data on aggregate per-capita consumption growth, inflation and yields (specifically, the 3-month and the 5-year nominal yields) are used to estimate the model. Different from other papers examining idiosyncratic risk and its effect on asset pricing (e.g., see Cogley (2002), Brav, Constantinides and Geczy (2002)), this paper didn't use household consumption data to construct the cross-sectional variance. Instead, I model the cross-sectional variance in the state vector and use yields to reverse engineer it. As warned in Cogley (2002), the presence of measurement error is a serious problem when using household consumption data, such as the consumer Expenditure Survey (CEX). Furthermore, CEX has only been available since 1980, which is too short a time to match other quarterly

data I have on growth, inflation and yields, which start from 1952.[2] Since cross-sectional variance is among the three factors in the pricing kernel that prices bonds, using observed yields of multi-period bonds would help to identify the cross-sectional variation that gives rise to them.

The estimated model shows that these three factors capture a number of features of observed yields. In particular, it implies that both real and nominal yield curves are upward sloping. This can be intuitively explained from a decomposition of the risk premium on long-term bonds into individual conditional covariances among three pricing factors and their expected future values.

For example, as in Piazzesi and Schneider (2006), the conditional covariance between inflation and expected future consumption growth is negative; that is, inflation is "bad news" for future consumption growth. Positive inflation surprises not only make nominal bonds have low real returns, but also forecast low future consumption growth. With the presence of idiosyncratic risk, this "bad news" effect of inflation is even amplified, because inflation is also positively correlated with expected future idiosyncratic variation, meaning that high inflation also forecasts high future idiosyncratic variation. The fact that nominal bonds pay off little precisely when the outlook of future worsens makes them unattractive assets to hold. Since long bonds pay off even less than short bonds when inflation - and hence bad news - arrives, agents require a term spread, or high yields, to hold them. This explains why nominal bonds command an inflation risk premium over real bonds, and more importantly, why the nominal yield curve is upward sloping.

Besides, the conditional covariance between consumption growth and expected future idiosyncratic risk is positive. This is important in understanding why the real yield curve is upward sloping. A high expected idiosyncratic risk in the future increases agents' desire for precautionary saving and thus raises the price of bonds today. This means bondholders' wealth increases in good times (marginal utility is low), and decreases in

[2]The PSID data, on the other hand, has a relatively longer sample, starting from 1968. However, one shortcoming of PSID is that data are available only for food consumption, and there are legitimate concerns about whether this is adequate for studying problems related to intertemporal substitution and self-insurance.

bad times (marginal utility is high). So they require a premium to offset this risk.

In calculating yields at all maturities, I use the technique of affine bond pricing. Under linearity and normality, the state space model leads to reduced solutions for bond yields; that is, yields of all maturities are affine functions of the pricing factors. Introducing affine bond pricing techniques improves the efficiency of the calculation and provides insight into the model. More specifically, the model produces over 97% of the volatility in quarterly nominal yields observed in the data, suggesting that changes in expected consumption growth, inflation and idiosyncratic risk are able to account for a vast part of yield dynamics.

The model contributes to the growing literature that examines the joint behavior of the yield curve and the macroeconomy. It incorporates the advantages of both the affine term structure factor models in finance and the consumption-based asset pricing models in macroeconomics.

The factor models in finance usually describe the yield curve dynamics using no-arbitrage conditions and then summarize the yield curve with a number of latent factors.[3] While being successful in matching the key features of the yield curve observed in the data, the factor models are generally lacking direct connections to the macroeconomic environment, without a characterization of the equilibrium in the economy.[4]

On the other hand, the consumption-based asset pricing model derives yields directly from the first order conditions of the agent's intertemporal optimization problem and has an intuitive characterization for the economy and hence the asset risk associated with it. However, the standard representative agent framework has not found much empirical support, with a hard time matching the stylized yields data.[5]

[3]For example, for the literature on latent or unobservable factor models, see Litterman and Scheinkman (1991), Duffie and Kan (1996), Dai and Singleton (2000); for later work on including macroeconomic variables as factors, see Ang and Piazzesi (2003).

[4]In Ang and Piazzesi (2003), macro variables are incorporated through a factor representation for the pricing kernel. There is no equilibrium characterization of the economy.

[5]For example, Backus, Gregory, and Zin (1989) examine a dynamic exchange economy with complete markets and find that the model can account for neither the sign nor the magnitude of the average term premium in the data. Similar results appear in Salyer (1990), Donaldson, Johsen, and Mehra (1990),

By relaxing the representative agent assumption and taking into account both aggregate and idiosyncratic risks, our analysis not only illuminates the latent factors in finance with an economic interpretation, but also improves the empirical performance of the consumption-based models.

There are also other papers along this line of combination. Most of them are within the representative agent framework but using non-standard preferences. Besides the 2006 paper of Piazzesi and Schneider mentioned above, Gallmeyer et al (2007) also use the Epstein-Zin recursive preference. More specifically, they combine recursive preferences with a stochastic volatility model for consumption growth and inflation. They find that when inflation is endogenous – related to growth and short rate through a Taylor rule, the model can provide a good fit for the yield curve. Within the same representative agent framework, Wachter (2006) uses a habit-persistence preference as in Campbell and Cochrane (1999) and finds that the negative correlation between surplus consumption and the short real rate leads to positive risk premium and an upward sloping yield curve.

The remainder of paper is organized as follows. Section 2 introduces the heterogenous agents setup and derives the implied asset pricing kernel in the presence of idiosyncratic risk. Section 3 models the dynamics of the pricing kernel. Section 4 estimates the model. Section 5 evaluates the model's implications for bond yields. Section 6 concludes.

2 Idiosyncratic Risk and the Pricing Kernel

I begin the analsysis of the yield curve by solving for equilibrium yields in an endowment economy with heterogenous agents. Following Constantinide and Duffie (1996), agents' preferences are identical. Heterogeneity lies in the fact that they are subject to uninsurable, persistent, idiosyncratic consumption shocks.

Each agent i has logarithmic expected utility:

$$E_0 \sum_{t=0}^{\infty} \beta^t \ln C_t^i \tag{1}$$

Boudoukh (1993).

where β is a subjective disoucnt factor.

Individual consumption, C_t^i, is determined by the product of an aggregate and an idiosyncratic stochastic process

$$\begin{aligned} C_t^i &= \delta_t^i C_t \\ where \;\; \delta_t^i &= \exp[\sum_{s=1}^{t}(\eta_s^i x_s - \frac{1}{2}x_s^2)] \end{aligned} \tag{2}$$

where C_t is aggregate consumption, δ_t^i is idiosyncratic component, and $\{\eta_s^i\}_{s=1}^t$ are idiosyncratic shocks, assumed to be standard normal $N(0,1)$ for all i and t.

The variable x_t is, by construction since it multiplies the shock η_t^i, the standard deviation of cross-sectional distribution of individual consumption growth relative to aggregate growth. To see this, note that

$$\begin{aligned} \ln(\frac{C_{t+1}^i/C_{t+1}}{C_t^i/C_t}) &= \ln(\frac{\delta_{t+1}^i}{\delta_t^i}) \\ &= \eta_{t+1}^i x_{t+1} - \frac{x_{t+1}^2}{2} \\ &\sim N(-\frac{x_{t+1}^2}{2}, x_{t+1}^2) \end{aligned}$$

Given this structure, Constantinides and Duffie (1996) prove that the individual consumption process in (2) is indeed an equilibrium consumption process, that is, the agent is exactly happy to consume $\{C_t^i\}$ without further trading in assets. Note that persistence of the idiosyncratic shocks is necessary for overcoming self-insurance[6]. Otherwise, the agent could smooth over idiosyncratic shocks by borrowing and lending. In that case the resulting equilibrium would closely approximate a complete market allocation. That's why early idiosyncratic risk papers found quickly how clever the consumers could be in getting rid of the idiosyncratic risks by trading the existing set of assets[7]. Constantinides and Duffie get around this problem by making the idiosyncratic risk permanent.

[6] The process for idiosyncratic consumption shocks δ_t^i is a martingale.

[7] For example, Telmer (1993) and Lucas (1994) calibrate economies in which agents face transitory but uninsurable income shocks. They conclude that agents are able to come close to the complete markets rule of complete risk sharing by borrowing and lending or by building up a stock of savings, even though they are allowed to trade in just one security in a frictionless market.

Under their no-trade theorem, the agent's private first-order condition for an optimal consumption-portfolio decison

$$1 = E_t[\beta(\frac{C^i_{t+1}}{C^i_t})^{-1} R_{t+1}] \tag{3}$$

holds exactly.

Furthermore, plugging the individual consumption process into the first-order condition, we can transform the individual Euler equation into an Euler equation that depends not only on the aggregate consumption growth but also on the cross-sectional variance of individual consumption growth:

$$1 = E_t[\beta(\frac{C_{t+1}}{C_t})^{-1} \exp(x^2_{t+1}) R_{t+1}] \tag{4}$$

To derive this, first substitute C^i_t with $\delta^i_t C_t$,

$$\begin{aligned} 1 &= E_t[\beta(\frac{C_{t+1}}{C_t})^{-1}(\frac{\delta^i_{t+1}}{\delta^i_t})^{-1} R_{t+1}] \\ 1 &= E_t[\beta(\frac{C_{t+1}}{C_t})^{-1} \exp(-(\eta^i_{t+1} x_{t+1} - \frac{x^2_{t+1}}{2})) R_{t+1}] \end{aligned}$$

Then use the law of iterated expectations $E[f(\eta x)] = E[E(f(\eta x)|x)]$. With η^i_t normal $(0, 1)$,

$$E[\exp(-(\eta^i_{t+1} x_{t+1} - \frac{x^2_{t+1}}{2})|x_{t+1}] = \exp(x^2_{t+1})$$

Therefore, we have

$$1 = E_t[\beta(\frac{C_{t+1}}{C_t})^{-1} \exp(x^2_{t+1}) R_{t+1}]$$

This leads to the real pricing kernel (or Stochastic Discount Factor, SDF) that prices all real assets under consideration

$$M_{t+1} \equiv \beta(\frac{C_{t+1}}{C_t})^{-1} \exp(x^2_{t+1}) \tag{5}$$

The random variable M_{t+1} represents essentially the date t prices of contigent claims that pay off one unit of consumption at $t+1$. In particular, a claim is expensive when the state it is contigent on is "bad". In a representative agent model, the bad state is the one in which future consumption growth is low. This effect is represented by the first term in the

pricing kernel. Heterogenous agents with uninsurable idiosyncratic risk introduces a new term - the state is also bad when the cross-sectional variance of individual consumption growth is high.

In order to speak to the empirical findings where we have returns in nominal terms, it is necessary to model nominal pricing kernel. Denote the nominal price index at time t as P_t, the Euler equation must hold for the real returns on nominal assets, therefore[8]

$$\frac{1^{\$}}{\mathrm{P}_t} = E_t[M_{t+1}\frac{R^{\$}_{t+1}}{\mathrm{P}_{t+1}}]$$

Rearranging terms and denoting the gross rate of inflation as $\Pi_{t+1} = \mathrm{P}_{t+1}/\mathrm{P}_t$, the Euler equation is reduced to

$$1^{\$} = E_t[\frac{M_{t+1}}{\Pi_{t+1}}R^{\$}_{t+1}] = E_t[M^{\$}_{t+1}R^{\$}_{t+1}]$$

Therefore, we have the nominal pricing kernel as

$$M^{\$}_{t+1} \equiv M_{t+1}/\Pi_{t+1} = \beta(\frac{C_{t+1}}{C_t})^{-1}(\Pi_{t+1})^{-1}\exp(x^2_{t+1}) \tag{6}$$

It represents the date t prices of contigent claims that pay off one dollar at $t+1$.

Taking logarithms, the log nominal pricing kernel is

$$m^{\$}_{t+1} = \ln\beta - \triangle c_{t+1} - \pi_{t+1} + x^2_{t+1} \tag{7}$$

where $\triangle c_{t+1} = \ln(C_{t+1}) - \ln(C_t)$ represents consumption growth, π_{t+1} is the rate of inflation, and x^2_{t+1} is the cross-sectional variance of individual consumption growth.

The price of a bond that pays one dollar n periods later, denoted as $P^{\$}_{nt}$, is therefore determined as the expected value of its payoff tomorrow weighted by the pricing kernel. Solving it forward suggests that it is determined by the expected values of future pricing kernel:

$$P^{\$}_{nt} = E_t(P^{\$}_{n-1,t+1}M^{\$}_{t+1}) = E_t(\prod_{i=1}^{n} M^{\$}_{t+i}) \tag{8}$$

[8] Throughout the paper, I use superscript '$' to denote nominal terms - payoffs denominated in dollars.

Assuming that the log pricing kernel is normally distributed, then by the property of log-normal distribution, we get log price

$$\begin{aligned} p^{\$}_{nt} &= E_t(p^{\$}_{n-1,t+1} + m^{\$}_{t+1}) + \frac{1}{2} var_t(p^{\$}_{n-1,t+1} + m^{\$}_{t+1}) \\ &= E_t(\sum_{i=1}^{n} m^{\$}_{t+i}) + \frac{1}{2} var_t(\sum_{i=1}^{n} m^{\$}_{t+i}) \end{aligned} \tag{9}$$

The yield for a continuously compounded n-period bond is then defined from the relation

$$y^{\$}_{nt} = -\frac{1}{n} \ln P^{\$}_{nt} = -\frac{1}{n} E_t(\sum_{i=1}^{n} m^{\$}_{t+i}) - \frac{1}{2n} var_t(\sum_{i=1}^{n} m^{\$}_{t+i}) \tag{10}$$

For a fixed date t, the (nominal) yield curve maps the maturity n of a bond to its yield $y^{\$}_{nt}$.

Equations (9) and (10) show that log prices and yields of bonds are determined by expected future consumption growth, inflation and idiosyncratic variation. Take the short rate for example

$$y^{\$}_{1t} = -\ln \beta + E_t(\triangle c_{t+1} + \pi_{t+1} - x^2_{t+1}) - \frac{1}{2} Var_t(\triangle c_{t+1} + \pi_{t+1} - x^2_{t+1}) \tag{11}$$

The effects of expected consumption growth and inflation are the same as that in a representative agent model – a high consumption growth in the future makes the agent save less today, thus interest rate increases; a high inflation in the future makes the agent prefer to consume today rather than tomorrow hence raises interest rate. Now with the presence of idiosyncratic consumption risk, a new term enters and affects yields in the opposite direction – an increase in expected future idiosyncratic risk leads to a decrease in yields. This is because an expected high idiosyncratic risk makes the agent more cautious about consuming today, that is, it increases his desire for precautionary savings. Thus bond prices increase and yields are reduced.

The variances of consumption growth, inflation and idiosyncratic risk have the same effect on yields – as the economy becomes more volatile, the agent saves more. The covaraines among three fundamental variables are also crucial. For example, the conditional covariance between consumption growth and inflation show that a negative covariance of consumption growth and inflation would lead to an increase in nominal rates. The reason

is, in that case, nominal bonds have low real payoffs exactly in bad times, which means that they can not provide a hedge against bad states, therefore, agents would demand higher nominal yeilds as compensation for holding them.

Therefore, a model of the term structure of interest rates is essentially a model describing the agents' beliefs about the time-series process of pricing kernel. That's exactly what the next section is about.

3 Model of the Fundamental Dynamics

This section outlines the time-series model for consumption growth ($\triangle c_{t+1}$), inflation (π_{t+1}) and idiosyncratic risk (x_{t+1}^2). Under linearity and normality, the model leads to reduced solutions for bond prices and yields, that is, bond prices of all maturities are exponential-affine functions of a small set of common state variables (or factors). Introducing affine bond pricing techniques improves the efficiency of the calculation and provides insight into the model.

3.1 The State-Space Representation

I model the dynamics of consumption growth, inflation and idiosyncratic risk in a state space representation, which has the following two main features.

First, consumption growth, inflation and idiosyncratic risk are correlated, and the correlation is imposed to allow for the maximal flexibility while still keep the model identifiable. The importance of the correlation among fundamentals can be seen from the covariance term in equations (8) and (11). This is consistent with Piazzesi and Schneider (2006), where they model consumption growth and inflation jointly in a state space model and find that the correlation between growth and inflation is critical; if inflation and consumption growth were independent, the nominal average yield curve would slope downward even with a recursive preference. Similarly, Gallmeyer et al (2007) examine the properties of the yield curve when inflation is exogenous – independent of growth, and when inflation is endogenous – related to growth and short rate through a Taylor

rule. They find that when inflation is exogenous, it is difficult to capture the slope of the historical average yield curve. In addition to growth and inflation, in this paper, idiosyncratic risk and its correlation with the other two fundamentals also play important roles. As for how the correlation is imposed, I follow Dai and Singleton (2000). They provide a complete specification analysis for affine term structure models. The parameterization here is similar to their Gaussian specification.

Second, idiosyncratic risk is directly modeled as a latent variable in the state equation, and is reverse engineered using the observation of yields. Unlike aggregate consumption growth and inflation, the data of which are handy, calculating idiosyncratic risk (cross-sectional variance of individual consumption growth) requires the use of household consumption data. The literature on using household data, e.g. Consumer Expenditure Survey data (CEX), suggests that measurement error is a serious problem (e.g., see Cogley, 2002). Besides, CEX has a short sample, starting from 1980, which is not long enough to match other quarterly data on growth, inflation and yields. Putting idiosyncratic risk in the state equation, the plan is to use observed yields to uncover the idiosyncratic risk that gives rise to them. For the yields used in estimation, I choose the short rate (3-month rate) and the 5-year rate, which are respectively the short end and the long end of the yield curve. Including long rate, which contains more forward information, would help to pin down the agent's long term forecasts.

The state space model is as follows:

$$\begin{aligned} S_{t+1} &= \mu_s + A(S_t - \mu_s) + C\varepsilon_{t+1} \\ Z_{t+1} &= \mu_z + DS_{t+1} + G\eta_{t+1} \end{aligned} \tag{12}$$

where $Z_{t+1} \equiv [\ \triangle c_{t+1} \quad \pi_{t+1} \quad y_{1t+1} \quad y_{20t+1}\ ]'$ is the observable vector containing consumption growth, inflation, 1-quarter yield and 20-quarter yield[9]. $S_{t+1} \equiv [\ s_{ct+1} \quad s_{\pi t+1} \quad x^2_{t+1}\ ]'$ is the state vector, with the first two elements governing expected consumption growth and expected inflation; and the third element as the cross-sectional variance of individual consumption growth. $\varepsilon_{t+1}$and η_{t+1} are uncorrelated standard normal $i.i.d$ innovations.

[9]The data are in quarters. Section 4 (estimation) describes the data in details.

A, C, D, G, μ_s and μ_z are parameter matrices and vectors specified as follows

$$\mu_s = \begin{bmatrix} 0 \\ 0 \\ \mu_{x^2} \end{bmatrix}; \; A = \begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}; \; C = \begin{bmatrix} c_{11} & 0 & 0 \\ c_{21} & c_{22} & 0 \\ c_{31} & c_{32} & c_{33} \end{bmatrix};$$

$$\mu_z = \begin{bmatrix} \mu_c \\ \mu_\pi \\ \alpha_1 \\ \alpha_{20} \end{bmatrix}; \; D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ & \beta_1' & \\ & \beta_{20}' & \end{bmatrix}; \; G = \begin{bmatrix} g_{11} & 0 \\ 0 & g_{22} \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The unconditional mean of the state vector is μ_s, with μ_{x^2} being the unconditional mean of cross-sectional variance (x_{t+1}^2) that needs to be estimated. Specifying the first two elements of μ_s as zero makes the unconditional mean of consumption growth and inflation in the measurement equation simply μ_c and μ_π. The companion matrix A and the cholesky decomposition of covariance matrix C are specified to allow for the maximal flexibility in the correlations of state variables, while still keep the model identifiable. The specification is similar to the canonical Gaussian representation in Dai and Singleton (2000). However, with their covariance matrix being diagonal, the conditional correlation between inflation and future consumption growth and the conditional correlation between idiosyncratic risk and future consumption growth or inflation are pre-assumed to be zero. I relax this by allowing a full covariance matrix. Yields in the measurement equation are affine functions of the state variables, with the intercept coefficients α_1, α_{20}(scalars) and slope coefficients β_1, β_{20} (3×1 vectors) that I'm going to describe below.

3.2 Affine Bond Pricing

Under normality and linearity of the state space model (12), bond prices defined in equation (8) can be reduced to exponential-affine functions of the state variables, which fall within the affine term structure framework. More precisely, bond prices are given by

$$P_{nt}^{\$} = \exp(\overline{\alpha}_n + \overline{\beta}_n' S_t) \tag{13}$$

where $\overline{\alpha}_n$ and $\overline{\beta}'_n$ can be found through the recursion

$$\begin{aligned}
\overline{\alpha}_{n+1} &= \overline{\alpha}_n + \ln\beta - \mu_c - \mu_\pi + (\overline{\beta}'_n + \phi_x)(I - A)\mu_s + \frac{1}{2}(\phi_c + \phi_\pi)GG'(\phi_c + \phi_\pi)' \\
&\quad + \frac{1}{2}[\overline{\beta}'_n + \phi_x - (\phi_c + \phi_\pi)D]CC'[\overline{\beta}'_n + \phi_x - (\phi_c + \phi_\pi)D]' \\
\overline{\beta}'_{n+1} &= (\overline{\beta}'_n + \phi_x - (\phi_c + \phi_\pi)D)A
\end{aligned} \tag{14}$$

starting with $\overline{\alpha}_0 = 0$ and $\overline{\beta}'_0 = \mathbf{0}_{1\times 3}$. $\phi_c = \begin{bmatrix} 1 & 0 & 0 & 0 \end{bmatrix}, \phi_\pi = \begin{bmatrix} 0 & 1 & 0 & 0 \end{bmatrix}$, and $\phi_x = [\ 0 \quad 0 \quad 1\ ]$ are the selecting vectors for $\triangle c$, π and x^2. These difference equations are derived by induction. Details are provided in the Appendix.

The n-period yield is therefore

$$y^{\$}_{nt} = -\frac{1}{n}\ln P^{\$}_{nt} = -\frac{1}{n}(\overline{\alpha}_n + \overline{\beta}'_n S_t) = \alpha_n + \beta'_n S_t \tag{15}$$

where $\alpha_n \equiv -\frac{1}{n}\overline{\alpha}_n$and $\beta'_n \equiv -\frac{1}{n}\overline{\beta}'_n$.

This gives the functional forms of α_1, β'_1 for the 1-quarter yield and α_{20}, β'_{20} for the 20-quarter yield in the measurement equation. Since yields are exact affine functions of the state variables, there are no pricing errors specified in matrix G.

Introducing affine bond pricing techniques improves the efficiency of calculation. Yields of all maturities can be calculated directly from equation (15). It also provides insight into the model. α_n is the intercept coefficient. β_n is the slope coeffcient, which we often refer to as "yields-factor loadings", since it loads the time-varying state variables into yields determination. For any fixed date t, the slope of the yield curve – yield spreads between n-period ($n > 1$) yield and short rate, $y^{\$}_{nt} - y^{\$}_{1t}$, is determined by both α_n and β_n. For any fixed maturity n, the time-series dynamics of the yield, however, is determined by β_n wieghted state variables. Therefore, α_n affects the level but not the dynamics of yields, while β_n affects both.

4 Estimation

The results of the previous section suggest that the process assumed for expected consumption growth, inflation and idiosyncratic risk is an important determinant of yields. In this section, I focus on estimating the state space model.

To estimate the state space model, I first apply the Kalman Filter algorithm to sequentially update the optimal forecasts of the state variables

$$\begin{aligned}\widehat{S}_{t+1|t} &= (I-A)\mu_s + A\widehat{S}_{t|t-1} + AP_{t|t-1}D'(DP_{t|t-1}D' + GG')^{-1}(Z_t - \mu_z - D\widehat{S}_{t|t-1}) \\ P_{t+1|t} &= AP_{t|t-1}A' - AP_{t|t-1}D'(DP_{t|t-1}D' + GG')^{-1}DP_{t|t-1}A' + CC'\end{aligned}$$

where $\widehat{S}_{t+1|t} = E[S_{t+1}|Z^t]$ is the conditional estimates of state variables based on information up till date t ; $P_{t+1|t} = E[(S_{t+1} - \widehat{S}_{t+1|t})(S_{t+1} - \widehat{S}_{t+1|t})']$ is the corresponding MSE of the estimates. The Kalman recursion is intialized by $\widehat{S}_{1|0}$, which denotes an estimate based on no observation, and its associated MSE $P_{1|0}$. It's reasonable to believe that the process for S_t is stationary, therefore, I set $\widehat{S}_{1|0}$ and $P_{1|0}$ at its uncondtional mean and variance. More precisely, $\widehat{S}_{1|0} = \mu_s$ and $P_{1|0} = AP_{1|0}A' + CC'$,which implies $vec(P_{1|0}) = [I_4 - A \otimes A]^{-1}vec(CC')$.

The state space model can then be estimated using maximum likelihood based on the assumption that the conditional density of Z_{t+1} on S_{t+1} is Gaussian with mean and variance as follows

$$Z_{t+1}|S_{t+1}, Z^t \sim N(\mu_z + \widehat{S}_{t+1|t},\ DP_{t+1|t}D' + GG')$$

Quarterly data on consumption growth, inflation and yields are used in estimation. The data are from Piazzesi and Schneider (2006). More specifically, aggregate consumption growth is measured using quarterly NIPA data on nondurables and services; inflation is measured using the corresponding price index; bond yields with maturities one year and longer are from CRSP Fama-Bliss discount data file; and the 1-quarter short rate is from CRSP Fama riskfree rate file.[10] The sample period is from the second quarter of 1952 to the last quarter of 2005.

The resulting maximum likelihood estimates, along with their estimated standard errors are displayed in Table 1. To reduce the dimension of free parameters, the uncondtional means for consumption growth and inflation are obtained from the sample means directly,

[10] CRSP: Center for Reseach in Security Prices. The actual bonds outstanding are usually with coupons, they construct the discount bonds data after extracting the term structure from a filtered subset of the available bonds.

that's why there is no standard errors reported. And the preference parameter β is set at a conventional value: 0.98. The data are in percent, for example, $\mu_c = 0.823$ represents a mean of 0.823 percent quarterly, that is an annualized mean of $0.823 \times 4 = 3.292$ percent for consumption growth.

To better understand the estimated dynamics, especially the process for idiosyncratic variation, I recover the smoothed estimates of the state variables from Kalman recursion. The Kalman smoother delivers a best estimate of the state conditional on all the data available. Specifically, it is obtained through the recursion

$$\widehat{S}_{t|T} = \widehat{S}_{t|t} + P_{t|t}A'P^{-1}_{t+1|t}(\widehat{S}_{t+1|T} - \widehat{S}_{t+1|t})$$

with the corresponding MSE

$$P_{t|T} = P_{t|t} + P_{t|t}A'P^{-1}_{t+1|t}(P_{t+1|T} - P_{t+1|t})(P_{t|t}A'P^{-1}_{t+1|t})'$$

Figure 1 plots the expected consumption growth implied by the model and the actual consumption growth data. The expected consumption growth is constructed using the smoothed estimates of the sate variables, specifically, $\widehat{\triangle c}_{t|T} = \mu_c + [\ 1 \quad 0 \quad 0\ ]\widehat{S}_{t|T}$. The red dotted lines are 2×standard error bounds from the corresponding MSE $P_{t|T}$. Figure 1 shows that the expected consumption growth series captures mainly the lower-frequency fluctuations in actual consumption growth. It explains around 33% of the standard deviation of actual consumption growth.

Similarly, Figure 2 plots the series for inflation. The expected inflation captures the fluctuations of actual inflation perfectly well. Indeed, the expected inflation explains 92% of the standard deviation of actual inflation, and almost all parts of the actual inflation series fall within the 2× standard error bands of expected inflation.

Next, Figure 3 plots the expected cross-sectional variation of individual consumption growth reverse engineered by the model. It shows two important features. First, the cross-sectional variation is highly persistent, with a first-order autocorrelation of 0.89. This confirms that idiosyncratic shocks must be persistent to have an effect on pricing implication. Second, the cross-sectional variation is large. The average of the cross-sectional variance is around 2.52%, which means that the cross-sectional standard deviation of

idiosyncratic risk is about $\sqrt{0.0252} = 0.1587$ or 15.87%, suggesting that we need high spread out of idisyncratic variation to account for asset pricing facts, especially in the case where the agent has a low risk aversion ($\gamma = 1$ in the logarithmic preference).[11] In fact, it is believed that there is a tradeoff between risk aversion and cross-sectional variation of idiosyncratic risk. For example, Cochrane (2005, chapter 21) shows that with high risk aversion, we do not need to specify highly volatile individual consumption growth, or dramatic sensitivity of the cross-sectional variance to the market return to explain the equity premium puzzle. But the risk aversion he probably needs are $\gamma = 25$ or even more. Therefore, in this paper, I didn't exploit this tradeoff and focus on the simple log-preference case. It would be an interesting exercise later to examine this trade-off by using a general power utility and higher risk aversion.

5 Implications for Yields

Based on model estimation, this section describes the model's implications for bond returns. I simulate model predicted yields by evaluating the affine pricing formula of section 3 at the Kalman smoother estimates of the state variables in section 4. The nominal yields at different maturities implied by the model are shown in Figure 4. The model produces a tremendous amount of the movements that we observe in the data and the difference between the two series is almost negligible. By this fit criterion, our reverse engineering exercise is a success. The rest of the section compares and explains the key moments in details.

[11] The literature on estimating/calculating cross-sectional variance of individual consumption growth gives various answers. For example, Deaton and Paxson (1994) report that the cross-sectional variance of log consumption within an age cohort rises from about 0.2 at age 20 to 0.6 at age 60, which means a standard deviation rises from 45% at age 20 to 77% at age 60, and that's a standard deviation of 1% per year. Using PSID data, Carroll (1992) estimates a value of 10% per year for permanent income shocks in his study of precautionary savings. Using CEX data, Cogley (2002) finds values for the quarterly cross-sectional standard deviation on the order of $35 - 40\%$ in the sample period 1980Q2 to 1994Q4.

5.1 Average Nominal Yields and the Bond Risk Premium

Table 2 reports the means of nominal yields. First, the model recovers the 3-month and 5-year yields that are used in estimation perfectly well. Second, the model predicts intermediate yields very close to the data. For example, the average yield on the 1-year bond in the model is equal to 5.48%, similar to the data average of 5.56%; the average yield on the 4-year bond is 6.064% in the model, approximately the same as 6.059% in the data.

Note that a standard representative agent CCAPM can not match the level of yields observed in the data. Instead, it predicts much higher yields. For example, Piazzesi and Schneider (2006) use a large subjective discount factor ($\beta = 1.005$) to reduce yields level predicted by a representative agent model with recursive preference; Xu (2008) finds that yields are on the order of 7.8% even after relaxing the rational-expectation assumption and taking into account the agent's robustness concern to model uncertainty. The key reason is that the representative agent framework focuses only on variation in aggregagte consumption and abstracts from idiosyncratic variation, and aggregate risk alone isn't big enough to make the agent save more, hence bond prices are too low and yields are too high.

Here with idiosyncratic consumption risk, the agent faces a lot more risk than is reflected in the variation of aggregate per capita consumption. This greater level of consumption risk makes the agent more cautious about consuming today; that is, it increases his desire for precautionary saving just in case bad things happen tomorrow. Thus individuals who must bear both aggregate and idiosyncratic risk will be willing to pay a higher price for transferring one unit of consumption from today to tomorrow, which makes bond prices increase and yields much smaller than one would predict using a representative agent model. This can also be seen from equation (11). Clearly, high expected idiosyncratic risk (x_{t+1}^2) leads to lower yields; and an increase in the volatility of idiosyncratic risk decreases yields as well.

Table 2 also demonstrates that the average yield curve on nominal bonds is upward sloping. For example, the average yield spread between the 3-month bond and the 5-year

bond is 0.99%, and the average spread between the 1-year bond and the 4-year bond is 0.58%. Figure 5 depicts the average yield curve predicted the model against the actual yield curve.

The intuitive explanation behind the positive slope can be understood in the conditional covariances captured by the joint dynamics of consumption growth, inflation and idiosyncratic risk.

Define $rx^{\$}_{n,t+1} = p^{\$}_{n-1,t+1} - p^{\$}_{nt} - y^{\$}_{1t}$ as the holding return on buying a n-period nominal bond at time t for $p^{\$}_{nt}$ and selling it at time $t+1$ for $p^{\$}_{n-1,t+1}$ in excess of the one-period short rate. Based on equation (9), the expected excess return can be derived as

$$\begin{aligned} E_t(rx^{\$}_{n,t+1}) &= -cov_t(m^{\$}_{t+1}, p^{\$}_{n-1,t+1}) - \frac{1}{2} var_t(p^{\$}_{n-1,t+1}) \\ or &= -cov_t(m^{\$}_{t+1}, E_{t+1} \sum_{i=1}^{n-1} m_{t+1+i}) - \frac{1}{2} var_t(p^{\$}_{n-1,t+1}) \end{aligned}$$

The covariance term on the right-hand side is the risk premium, while the variance term is due to Jensen's inequality. The risk premium on nominal bonds is positive when the pricing kernel and long bond prices are negatively correlated, or when the autocorrelation of the pricing kernel is negative. In this case, long bonds are less attractive than short bonds, because their payoffs tend to be low when the pricing kernel is high (marginal utility is high). Over long samples, the average excess return on a n-period bond is approximately equal to the average spread between the n-period yield and the short rate[12]. This means that the yield curve is on average upward sloping if the risk premium is positive on average.

In this model, the pricing kernel is determined by consumption growth, inflation and idiosyncratic risk. Plugging equation (7) into the covariance term, I decompose risk premium into individual conditional covariances between consumption growth, inflation,

[12] To see this, we can write the excess return as

$$\begin{aligned} p^{\$}_{n-1,t+1} - p^{\$}_{n,t} - y^{\$}_{1,t} &= n y^{\$}_{nt} - (n-1) y^{\$}_{n-1,t+1} - y^{\$}_{1t} \\ &= y^{\$}_{nt} - y^{\$}_{1t} - (n-1)(y^{\$}_{n-1,t+1} - y^{\$}_{nt}) \end{aligned}$$

For large n and a long sample, the difference between the average $(n-1)$-period yield and the average n-period yield is approximately zero.

idiosyncratic risk and their expected future values. More specifically

$$\begin{aligned} Risk\ Premium &= -cov_t(m^{\$}_{t+1}, E_{t+1}\sum_{i=1}^{n-1} m_{t+1+i}) \\ &= -cov_t(\triangle c_{t+1} + \pi_{t+1} - x^2_{t+1},\ E_{t+1}\sum_{i=1}^{n-1} \triangle c_{t+1+i} + \pi_{t+1+i} - x^2_{t+1+i}) \end{aligned}$$

Figure 6 plots the individual terms that determine the risk premium as a function of maturity. The terms that contribute to a positive risk premium are those with positive signs. Some of them are familar terms from the representative agent case like in Piazzesi and Schneider (2006). For example, the minus covariance between inflation and expected future consumption growth, $-cov_t(\pi_{t+1}, E_{t+1}\sum_{i=1}^{n-1} \triangle c_{t+1+i})$. This term is positive, because of the minus sign and the fact that positive inflation surprises forecast lower future consumption growth. In this case, inflation is bad news for consumption growth and nominal bonds have low payoffs exactly when inflation, and hence bad news, arrives. Since the payoffs of long-term bonds are affected even more than those of short bonds (note that the covariance term is increasing in maturity), agents require a premium, or high yields, to hold them.

With the presence of idiosyncratic risk, the "bad news" effect of inflation is amplified. The covariance term between inflation and expected future idiosyncratic variation, $cov_t(\pi_{t+1}, E_{t+1}\sum_{i=1}^{n-1} x^2_{t+1+i})$, shows that it is positive and has the biggest magnitude in all cross-covariance terms. Therefore, inflation is not only bad news for aggregate consumption growth, but also bad news for idiosyncratic risk. Surprise inflation lowers the payoff on nominal bonds and forecasts high future idiosyncratic variation. The fact that nominal bonds pay off little precisely when the outlook of the economy worsens makes them unattractive assets to hold.

Another important term is the positive conditional covariance between consumption growth and expected future idiosyncratic variation. This is important in understanding why the real yield curve is upward sloping. A high expected idiosyncratic risk in the future increases agents' desire for precautionary saving and thus raises the price of bonds today. This means bondholders' wealth increases in good times (marginal utility is low), and decreases in bad times (marginal utility is high). So they require a premium to offset

this risk.

The last two candidates that contribute a positive risk premium for long-term bonds are the negative covariance between consumption growth and expected future inflation, and the positive covariance between idiosyncratic risk and expected future inflation. The idea is similar. When expected future inflation is high, agents would prefer to consume today rather than tomorrow, thus bond prices are low, and this happens exactly when consumption growth is low or idiosyncratic risk is high. Thus long bonds are not good assets to hedge against bad states and investors require a premium to hold them.

The rest four terms all have negative signs. Especially, both idiosyncratic risk and inflation show highly positive autocorrelations.

From the affine term structure model's point of view, the upward slope can also be seen from the factor loadings across the yield curve. Equation (15) shows that yields are affine functions of the state variables: $[\ s_c \quad s_\pi \quad x^2\ ]'$. The effect of each state variable on the yield curve is determined by the loadings β_n that the state space model assigns on each yield of maturity n according to recursion (14). Figure 7 plots these loadings as a function of yield maturity. The coefficient of the third factor – idiosyncratic risk – is upward sloping, while the coefficients of S_c and S_π are decreasing. However, since the unconditional mean of the state vector is $\mu_s = \begin{bmatrix} 0 & 0 & \mu_{x^2} \end{bmatrix}$, the only loading that affects the average slope of the yield curve is the loading on x^2, and idiosyncratic risk is therefore corresponding to the "slope" factor in most affine term structure literature.

One last thing about the average nominal yield curve is that the data show a steep incline from the 3-month maturity to the 1-year maturity, while the model is hard to capture that. A potential explanation for the steep incline in the data is liquidity issues that may depress short Treasury bills relative to others. However, this liquidity premium is not present in this model.

5.2 Volatility and Autocorrelation of Nominal Yields

Table 3 reports the standard deviation of nominal yields across maturities. The model produces a large amount of volatility observed in the data. For example, the model

implies that the standard deviation for the 1-year yield is 2.84 percent, about 97% of the 2.92 percent in the data. For the 4-year yield, the standard deviation implied by the model is 2.77 percent, almost the same as the 2.78 percent in the data. Since yields are determined by affine functions of the state variabes, this suggests that changes in expected consumption growth, inflation and idiosyncratic variation are able to account for a vast part of nominal yields volatility. This is in line with the common finding in multifactor affine term structure literature. For example, Litterman and Scheinkman (1991) use a principal components approach and find that three factors – extracted from yields themselves – can explain well over 95% of the variation in weekly changes to U.S Treasury bond prices, for maturities up to 18 years.

Another feature of the model is that it does a good job in matching the high auto-correlaiton of yields at all maturities. Table 4 reports the first-order autocorrelation of yields. The autocorrelation in the 1-year yield is 95.1 percent, and the model produces 94.4 percent. For the 4-year yield, the autocorrelaiton in the model is 96.2 percent and only slightly below the 96.4 percent in the data. Also, the model captures the feature that long yields are more persistent than short yields, just as that in the data.

5.3 Predictions for Longer Nominal Yields and Real Yields

Table 5 and 6 report the model's predictions for nominal yields with longer maturities and real yields. The reason is to do robust check on the model's performance for explaining yields that are not used in estimation. As shown above, the model successfully captures the key features of nominal yields up to 5 years, although we only use the information on 3-month and 5-year yields. Here I do more. First, I check the model-implied nominal yields with much longer maturities – 6-year to 10-year. Table 5 shows that all key features are preserved – the average nominal yield curve keeps upward sloping; yields are still volatile; yields are highly autocorrelated, with long yields are more persistent than short yields.[13]

I also check the model-implied real yields and the yield spreads between nominal and

[13]Note that the CRSP Fama-Bliss Discount Bond File only contains bonds data with maturities up to 5 years. That's why there is no data reported for direct comparison.

real bonds. Table 6 reports the results. It shows that the average real yield curve slopes upward. This can be understood from the risk premium decomposition shown in Figure 6 as well. Abstracting from those terms with either inflation or expected future inflation[14], the rest are the terms that consist of the excess holding return on a multi-period real bond. The term that contributes to a positive risk premium on real bonds is the conditional covariance between consumption growth and expected future idiosyncratic variation. It is positive and outweighs other negative terms that are mainly due to the positive autocorrelations of consumption growth and idiosyncratic variation. The intuition is that a high expected idiosyncratic risk in the future increases agents' desire for precautionary saving and thus raises the price of bonds today. This causes bondholders' wealth to increase in good times, and decrease in bad times. So they require a premium to offset this risk.

Furthermore, the spreads between nominal and real yields are positive. This is mostly due to the impact of expected inflation (an average of 3.731 percent annualy). However, the Fisher relation doesn't explain all of the spreads. Nominal yields also incorporate a positive inflation risk premium. For example, equation (11) shows that the covariance between inflation and consumption and the covariance between inflation and idiosyncratic risk also determine nominal yields, and this inflation risk premium is positive because nominal bonds have low payoffs when consumption growth is low (since $cov_t(\triangle c_{t+1}, \pi_{t+1}) < 0$), or idiosyncratic risk is high (since $cov_t(x_{t+1}^2, \pi_{t+1}) > 0$).

6 Conclusion

This paper extends the standard consumption-based asset pricing model with power utility to a heterogeneous-agents framework. The key assumption is that agents are subject to both aggregate and idiosyncratic risks. Following Constantinides and Duffie (1996), this leads to a pricing kernel that depends not only on aggregate per capita consumption growth and inflation, but also on the cross-sectional variance of individual consumption growth. The dynamics of the pricing kernel is modeled in a state-space

[14]Note that from equation (5), the log real pricing kernel is $m_{t+1} = \ln \beta - \triangle c_{t+1} + x_{t+1}^2$.

representation that allows for the maximal correlations among pricing factors. Under linearity and normality, the model falls within the affine term structure framework, that is, bond yields of all maturities are affine functions of the state variables. Cross-sectional variance of individual consumption growth is modeled directly as a state variable in the state equation, and is reverse engineered using the observation of yields.

The maximum likelihood estimation of the state-space model using quarterly data on per capita consumption growth, inflation and 3-month and 5-year nominal yields shows that the model can account for many features of the nominal term structure of interest rates in the US. More specifically, it captures not only the level, but also the slope and volatility of the yield curve. The intuitive explanation behind the positive slope of the nominal yield curve is that inflation is bad news for both consumption growth and idiosyncratic variation. A positive surprise inflation not only lowers the real return on a bond, but also is associated with lower future consumption growth and higher idiosyncratic risk. In such a situation, bondholders' wealth decreases just as their marginal utility rises, so they require a premium to offset this risk.

Although the model focuses on examining the behavior of bonds, it can be extended to a broader class of assets, such as equity or even exchange rate. For example, we can specify a seperate exogenous process for the dividend growth like that in DeSantis (2007), or we can follow Wachter (2006) to treat the market portfolio as equivalent to aggregate wealth and the dividend equal to aggregate consumption[15]. Under either specification, the price of an equity is determined by the same pricing kernel as in equation (5). Examining the implication for exchange rates behavior would also be very interesting. More specifically, it could be related to the most recent affine term structure models of currency, for example, see Backus et al. (2001). The idea is that we can specify a process for each of the domestic pricing kernel and the foreign pricing kernel, and no-arbitrage ensures that these two kernels can be connected through exactly the change in exchange rate.

[15]In that case, an equity at period t is an asset that pays the endowment C_{t+n} in n periods. Therefore there is no need to introduce an additional variable into the problem.

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A Affine Pricing Recursion

To derive the recursion in equation (14), first start with a one-period bond

$$\begin{aligned}
P_{1t}^{\$} &= E_t(M_{t+1}^{\$}) \\
&= \exp\{E_t(m_{t+1}^{\$}) + \frac{1}{2}Var_t(m_{t+1}^{\$})\} \\
&= \exp\{E_t[\ln\beta - \triangle c_{t+1} - \pi_{t+1} + x_{t+1}^2] + \frac{1}{2}Var_t[.]\} \\
&= \exp\{E_t[\ln\beta - \phi_c Z_{t+1} - \phi_\pi Z_{t+1} + \phi_x S_{t+1}] + \frac{1}{2}Var_t[.]\} \\
&= \exp\{E_t[\ln\beta - (\phi_c + \phi_\pi)(\mu_z + DS_{t+1} + G\eta_{t+1}) + \phi_x S_{t+1}] + \frac{1}{2}Var_t[.]\} \\
&= \exp\{E_t[\ln\beta - \mu_c - \mu_\pi + (\phi_x - (\phi_c + \phi_\pi)D)((I-A)\mu_s + AS_t + C\varepsilon_{t+1}) \\
&\quad -(\phi_c + \phi_\pi)G\eta_{t+1}] + \frac{1}{2}Var_t[.]\} \\
&= \exp\{\ln\beta - \mu_c - \mu_\pi + \phi_x(I-A)\mu_s \\
&\quad +\frac{1}{2}[\phi_x - (\phi_c + \phi_\pi)D]CC'[\phi_x - (\phi_c + \phi_\pi)D]' \\
&\quad +\frac{1}{2}(\phi_c + \phi_\pi)GG'(\phi_c + \phi_\pi)' \\
&\quad +(\phi_x - (\phi_c + \phi_\pi)D)AS_t\}
\end{aligned}$$

Matching coefficients gives $\overline{\alpha}_1 = \ln\beta - \mu_c - \mu_\pi + \phi_x(I-A)\mu_s + \frac{1}{2}[\phi_x - (\phi_c+\phi_\pi)D]CC'[\phi_x - (\phi_c+\phi_\pi)D]' + \frac{1}{2}(\phi_c+\phi_\pi)GG'(\phi_c+\phi_\pi)'$ and $\overline{\beta}_1' = (\phi_x - (\phi_c+\phi_\pi)D)A$. Note that the last equality relies on the assumption that $E_t(\varepsilon_{t+1}) = 0$ and $Var_t(\varepsilon_{t+1}) = I$.

Suppose the price of an n-period bond satisfies $P_{nt}^{\$} = \exp(\overline{\alpha}_n + \overline{\beta}_n' S_t)$, next we show that the exponential form also applies to the price of a $n+1$ period bond

$$
\begin{aligned}
P_{n+1,t}^{\$} &= E_t(P_{n,t+1}^{\$} M_{t+1}^{\$}) \\
&= E_t\{\exp(\overline{\alpha}_n + \overline{\beta}_n' S_{t+1})\exp(m_{t+1}^{\$})\} \\
&= E_t\{\exp[\overline{\alpha}_n + \ln\beta - \mu_c - \mu_\pi + (\overline{\beta}_n' + \phi_x - (\phi_c+\phi_\pi)D)S_{t+1} \\
&\quad -(\phi_c+\phi_\pi)G\eta_{t+1}]\} \\
&= E_t\{\exp[\overline{\alpha}_n + \ln\beta - \mu_c - \mu_\pi + (\overline{\beta}_n' + \phi_x - (\phi_c+\phi_\pi)D) \\
&\quad ((I-A)\mu_s + AS_t) + (\overline{\beta}_n' + \phi_x - (\phi_c+\phi_\pi)D)C\varepsilon_{t+1} \\
&\quad -(\phi_c+\phi_\pi)G\eta_{t+1}]\} \\
&= \exp\{\overline{\alpha}_n + \ln\beta - \mu_c - \mu_\pi + (\overline{\beta}_n' + \phi_x)(I-A)\mu_s \\
&\quad +\frac{1}{2}[\overline{\beta}_n' + \phi_x - (\phi_c+\phi_\pi)D]CC'[\overline{\beta}_n' + \phi_x - (\phi_c+\phi_\pi)D]' \\
&\quad +\frac{1}{2}(\phi_c+\phi_\pi)GG'(\phi_c+\phi_\pi)' \\
&\quad +[\overline{\beta}_n' + \phi_x - (\phi_c+\phi_\pi)D]AS_t\}
\end{aligned}
$$

Matching coefficients results in the recursion in equation (14).

Table 1: Maximum-Likelihood Estimates of the State-Space Model

	μ	A			C			G	
$\triangle c$	0.823	0.954	-	-	0.048	-	-	0.446	-
		(0.019)			(0.01)			(0.019)	
π	0.927	−0.540	0.796	-	−0.172	0.073	-	-	−0.214
		(0.243)	(0.04)		(0.021)	(0.011)			(0.018)
x^2	1.918	2.247	0.614	0.983	−0.084	−0.158	0.197	-	-
	(0.718)	(2.008)	(0.54)	(0.011)	(0.057)	(0.043)	(0.024)	-	-

Note: This table contains the maximum-likelihood estimates for the state-space model

$$\begin{aligned} S_{t+1} &= \mu_s + A(S_t - \mu_s) + C\varepsilon_{t+1} \\ Z_{t+1} &= \mu_z + DS_{t+1} + G\eta_{t+1} \end{aligned}$$

I estimate the model using quartely data on consumption growth, inflation, 3-month short rate and 5-year rate over the sample period 1952Q2-2005Q4. The numbers in parentheses are maximum-likelihood asymptotic standard errors computed from the outer-product of the scores of the log-likelihood function.

Table 2 : Average Nominal Yields

Maturity	Model	Data
1	5.15	5.15
4	5.48	5.56
8	5.77	5.76
12	5.95	5.93
16	6.06	6.06
20	6.14	6.14

Note: This table reports the means of nominal yields from the model and from the data. Yields are in annual percentages. Maturity is in quarters. The sample period is 1952Q2-2005Q4.

Table 3 : Volatility of Nominal Yields

Maturity	Model	Data
1	2.92	2.92
4	2.84	2.92
8	2.80	2.88
12	2.79	2.81
16	2.77	2.78
20	2.74	2.74

Note: This table reports the standard deviations of nominal yields from the model and from the data. Numbers are in annual percentages. Maturity is in quarters. The sample period is 1952Q2-2005Q4.

Table 4 : Autocorrelation of Nominal Yields

Maturity	Model	Data
1	0.936	0.936
4	0.944	0.951
8	0.953	0.960
12	0.958	0.963
16	0.962	0.964
20	0.965	0.965

Note: This table reports the first-order autocorrelations of nominal yields from the model and from the data over the period 1952Q2-2005Q4.

Table 5 : Nominal Yields with Longer Maturities

Maturity	Mean	Standard Deviation	Autocorrelation
24	6.190	2.708	0.965
28	6.221	2.671	0.966
32	6.238	2.631	0.966
36	6.244	2.588	0.966
40	6.246	2.543	0.967

Note: This table reports the moments of nominal yields with longer maturities implied by the model. Numbers are in annual percent. Maturity is in quarters. The CRSP data contains only yields up to 5-year maturity, that's why there is no data reported for direct comparison.

Table 6 : Real Yields

Maturity	Mean	Standard Deviation	Autocorrelation	Nominal-Real Spreads
1	1.366	1.950	0.895	3.782
4	1.654	1.964	0.922	3.830
8	1.907	1.990	0.937	3.862
12	2.072	2.006	0.944	3.876
16	2.185	2.013	0.947	3.879
20	2.264	2.012	0.950	3.876

Note: This table reports the moments of real yields implied by the model. Numbers are in annual percent. Maturity is in quarters.

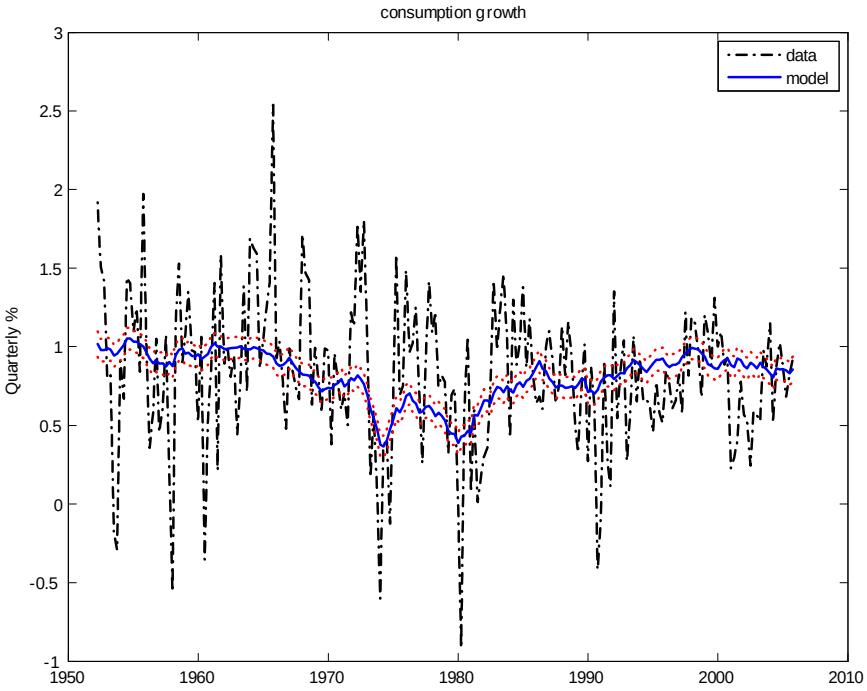

Figure 1: Expected consumption growth implied by the model and actual consumption growth in the data. The expected consumption growth is constructed using the smoothed estimates of the state vector. That is, $\widehat{\triangle c}_{t|T} = \mu_c + [\ 1 \quad 0 \quad 0\ ]\widehat{S}_{t|T}$. The red dotted lines are 2×standard error bounds from the corresponding MSE $P_{t|T}$.

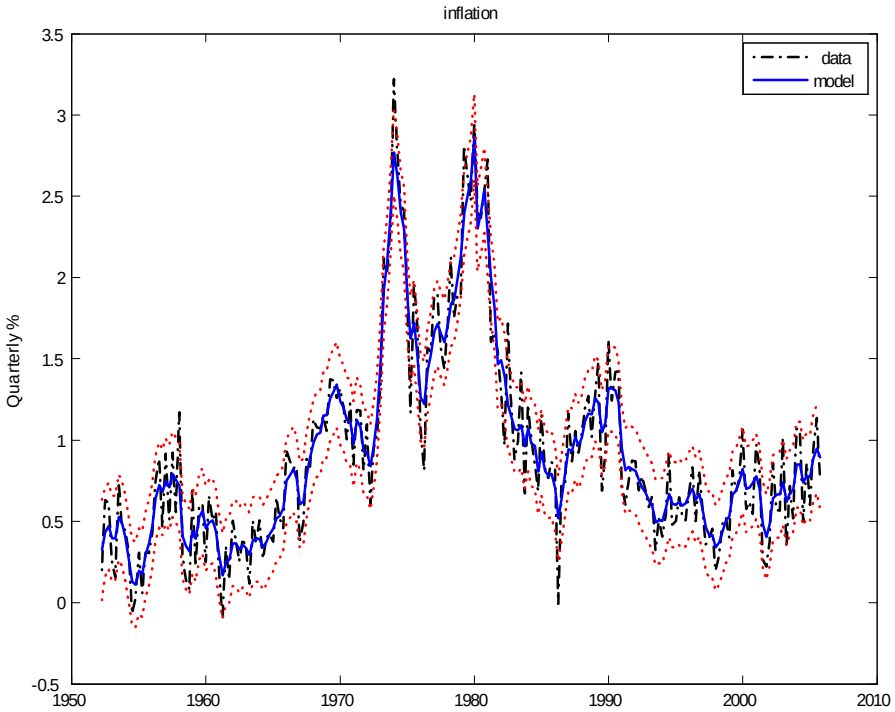

Figure 2: Expected inflation implied by the model and actual inflation in the data. The expected inflation is constructed using the smoothed estimates of the state vector. That is, $\widehat{\pi}_{t|T} = \mu_\pi + [\ 0 \quad 1 \quad 0\ ]\widehat{S}_{t|T}$. The red dotted lines are 2×standard error bounds from the corresponding MSE $P_{t|T}$.

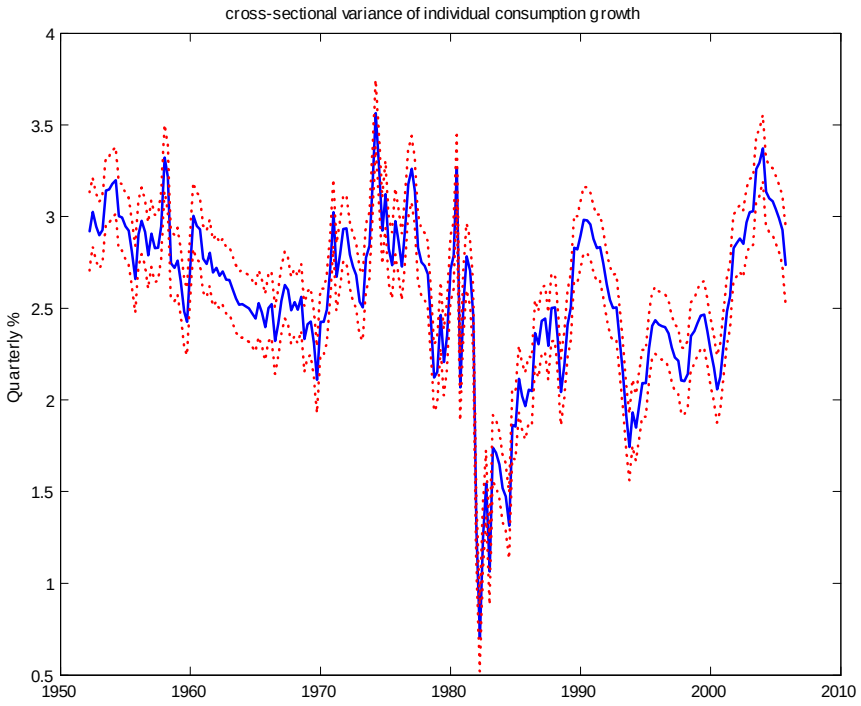

Figure 3: Expected cross-sectional variance of individual consumption growth implied by the model. It is constructed using the smoothed estimates of the state vector. That is, $\widehat{x^2}_{t|T} = [\ 0 \quad 0 \quad 1\ ]\widehat{S}_{t|T}$. The red dotted lines are 2×standard error bounds from the corresponding MSE $P_{t|T}$.

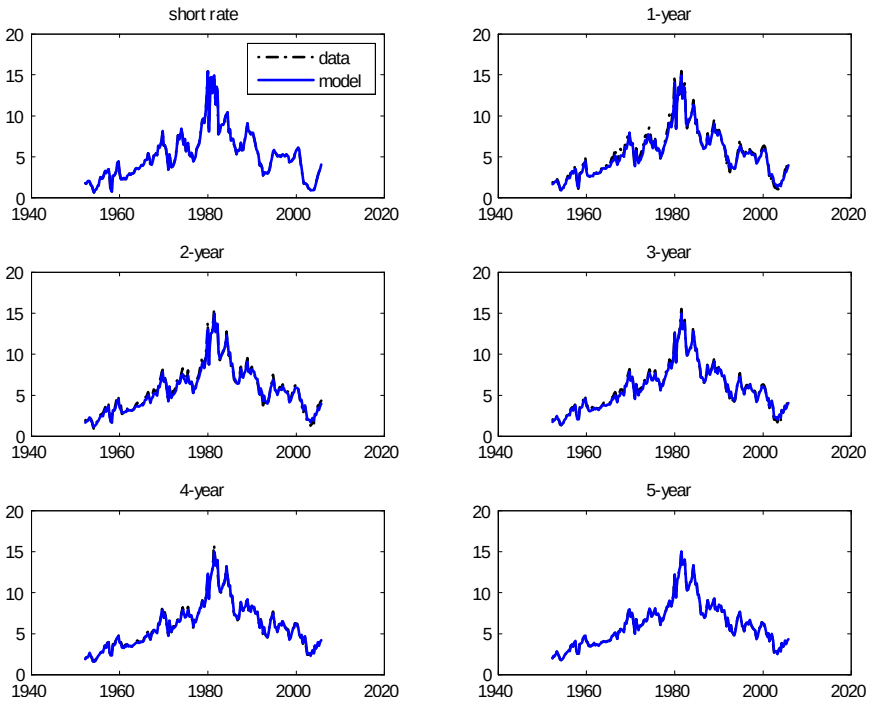

Figure 4: Time series dynamics of nominal yields at different maturities implied by the model and in the data. Numbers are in annual percent.

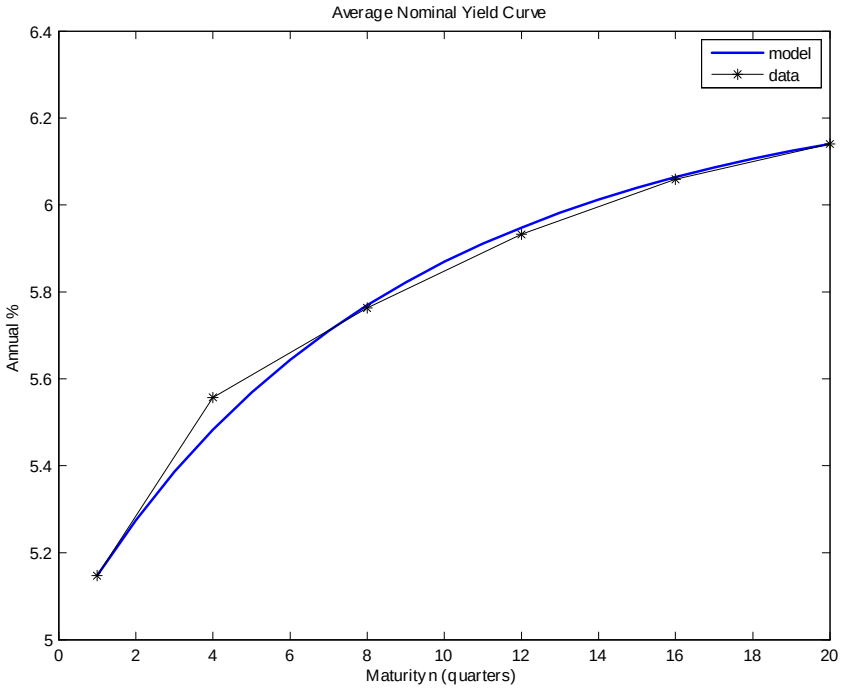

Figure 5: Average nominal yield curve. Numbers are in annual percent.

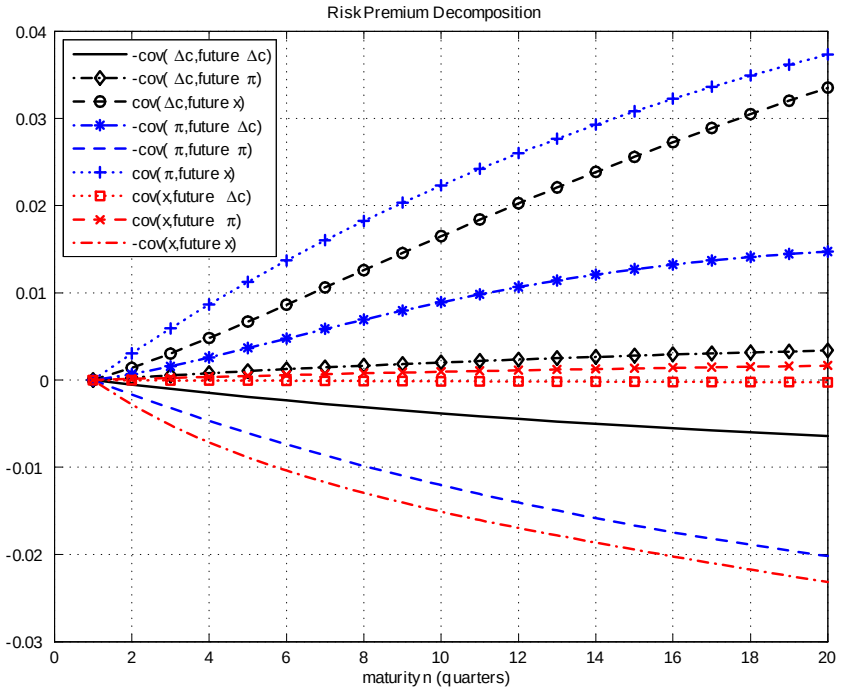

Figure 6: The decomposition of the risk premium on a multi-period bond excess holding return into individual conditional covariance terms between consumption growth, inflation, idiosyncratic risk and their expected future values.

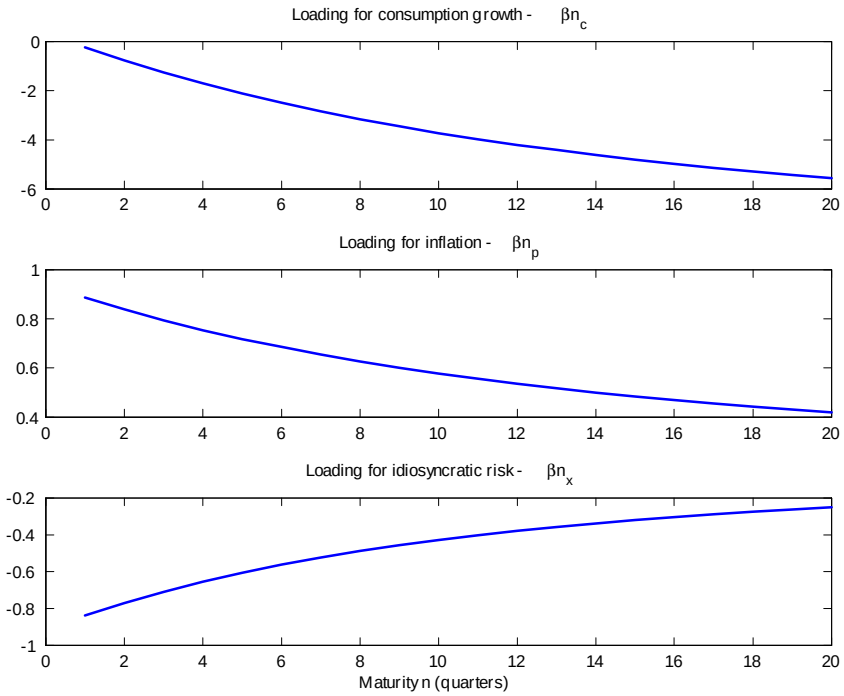

Figure 7: The slope coefficient β_n in the affine yields formula as a function of maturity n. The first element of β_n loads consumption growth into yields determination; the second element loads inflation, the third element loads idiosyncratic risk.